

UNIVERSITE DE GENEVE

Département de
physique théorique

FACULTE DES SCIENCES

Professeur J.-P. ECKMANN

NOYAUX IRRÉDUCTIBLES ET ÉTATS LIÉS DANS $\lambda P(\phi)_2$

THESE

présentée à la Faculté des sciences
de l'Université de Genève
pour obtenir le grade de
docteur ès sciences, mention physique

par

Hans KOCH

de

Schönenwerd (SO)

THESE No 1919

GENEVE

imprimerie de la Section de Physique

1979

UNIVERSITE DE GENEVE

Département de
physique théorique

FACULTE DES SCIENCES

Professeur J.-P. ECKMANN

NOYAUX IRRÉDUCTIBLES ET ÉTATS LIÉS DANS $\lambda P(\phi)_2$

THESE

présentée à la Faculté des sciences
de l'Université de Genève
pour obtenir le grade de
docteur ès sciences, mention physique

par

Hans KOCH

de

Schönenwerd (SO)

THESE No 1919

GENEVE

imprimerie de la Section de Physique

1979

La faculté des sciences, sur le préavis de Messieurs J.-P. ECKMANN, professeur assistant et directeur de thèse, M. GUENIN, professeur ordinaire (Dpt de physique théorique - Genève) et R. SENEOR, docteur ès sciences (Ecole Polytechnique - Palaiseau, France) autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

Genève, le 26 juin 1979

Le Doyen :
Bernard LEVRAT

Thèse No 1919



REMERCIEMENTS

Je tiens à remercier ici, toutes les personnes qui ont participé à mon éducation de physicien et tout particulièrement mon directeur de thèse, Jean-Pierre Eckmann, à qui je dois bien plus que je ne saurais exprimer en quelques mots.

Mes remerciements vont également à :

- Messieurs M. Guenin et R. Sénéor qui ont bien voulu accepter la responsabilité de juré de thèse.
- Andreas, Denis, François, Max, Michel, Pierre, Ricardo, Terje, Werner et tous les autres membres du département de physique théorique pour avoir su créer une atmosphère amicale à l'institut.
- Monsieur J. Glimm pour m'avoir indiqué une erreur dans une version précédente de ce travail.
- Mademoiselle F. Nicole pour avoir dactylographié le manuscrit de ce travail.

ABSTRACT

We analyze the mass spectrum below the two particle threshold for weakly coupled $P(\phi)_2$ quantum field models. Criteria for the existence of a two particle bound state and an asymptotic expansion for its mass are given in terms of the coefficients of the interaction polynomial. The analysis is based on analyticity properties and perturbation theory for n -particle irreducible kernels. The same methods are applied to the ϕ^4 theory with strong external field $\mu\phi$ to prove the existence of exactly one two particle bound state.

INTRODUCTION ET RESUME

Aujourd'hui encore, la théorie quantique des champs est l'outil le mieux adapté à la description des particules élémentaires et leurs interactions.

Un exemple de théorie des champs, celle ayant le plus de succès à son actif, est l'électrodynamique quantique. Elle a débuté il y a une cinquantaine d'années, par les travaux de Dirac.

Ses succès d'une part et son état mathématique peu satisfaisant d'autre part, ont conduit à la théorie constructive des champs. Celle-ci a débuté, il y a une douzaine d'années environ et a pour objectif, d'exhiber une théorie satisfaisante du point de vue mathématique et qui justifierait les calculs formels de l'électrodynamique quantique en particulier.

Le cadre de la théorie constructive des champs est donnée par un ensemble d'axiomes, que l'on espère être vérifiés dans les théories physiques. Ces axiomes entraînent quelques conséquences intéressantes telles que les théorèmes P.C.T., spin et statistiques, symétrie du crossing. Mais jusqu'à présent, personne ne sait s'il existe des théories non triviales, satisfaisant à ces axiomes.

Au vu de la difficulté des problèmes posés, il est raisonnable de résoudre tout d'abord d'autres problèmes, similaires mais plus simples. L'une des simplifications consiste à considérer un monde à deux ou trois dimensions d'espace-temps. Grâce à cette restriction, les progrès réalisés ces dernières années permettent l'étude et la compréhension de différents modèles. C'est en particulier le programme Euclidien proposé par Symanzik et réalisé par Nelson qui a rendu possible ces développements ainsi que leur connection avec la mécanique statistique.

L'exemple le plus simple et le plus connu $\lambda P(\phi)_2$ est dû à Glimm et Jaffe. Il décrit un champ scalaire de bosons dans deux dimensions d'espace-temps, faiblement auto-couplé de la manière suivante

$$*) \quad (\partial_t^2 - \partial_x^2 + m^2) \varphi(t, x) = -\lambda : P'(\varphi) : (t, x)$$

où P est un polynôme de la forme :

$$**) \quad P(\varphi) = \sum_{j=0}^{2n} c_j \varphi^j, \quad c_{2n} > 0.$$

$P(\phi) = \phi^m$, par exemple, décrit une interaction directe : "k particules donnent m-k particules" $0 < k < m$. Les puissances impaires ϕ^j et en particulier $j = 1$ décrivent l'influence d'un champ extérieur.

Dans une théorie libre, le spectre de masse est constitué de deux points 0 et m , correspondant à l'état de vide et l'état à une particule, et d'un spectre continu de $2m$ à l'infini, correspondant aux états à plusieurs particules. Dans une théorie avec interaction le spectre n'est plus nécessairement identique à celui d'une théorie libre. Même avec des interactions faibles on peut s'attendre à des valeurs propres supplémentaires qui représentent de nouvelles espèces de particules élémentaires, appelées états liés.

Du point de vue technique, le cas $P(\phi) = \phi^4$ est le plus simple à discuter; le spectre de masse et la complétude asymptotique sont connus pour des énergies inférieures à $3m$. Le même résultat est vrai pour la plupart des autres polynômes pairs, à la différence près que quelques unes de ces interactions produisent un état lié de deux particules initiales [DE]. Leur (non-) existence peut être déterminée par l'étude du noyau de Bethe-Salpeter.

Le but de ce travail est d'étendre l'analyse du spectre de masse à toutes les interactions polynômiales de la forme **). L'abandon de la symétrie $\phi \rightarrow -\phi$ complique considérablement la chose. En plus des critères d'existence ou de non-existence d'un état lié (qui ne dépend plus seulement du noyau de Bethe-Salpeter) nous donnons aussi des développements asymptotiques pour les

masses et les renormalisations du champ.

Ce travail repose essentiellement sur la convergence de la "cluster expansion" [GJ I], [GJS]. Elle nous fournit des fonctions de Schwinger, C^∞ dans la constante de couplage (petite et non-négative) présentant suffisamment de régularité pour pouvoir construire les noyaux n-particules irréductibles nécessaires (voir Chapitre II). Ces noyaux sont étudiés dans les Chapitres III et IV. Leurs propriétés d'analyticité, combinées avec des équations du type Bethe-Salpeter, impliquent une structure méromorphe pour les fonctions à n points. Les fonctions à deux et à quatre points ainsi que la connection entre leurs pôles et le spectre de masse sont discutées dans les Chapitres V, VI et VII. Les résultats concernant les états liés sont énoncés et démontrés au Chapitre VIII. Finalement, le dernier chapitre est consacré à une application à la théorie ϕ^4 avec champ extérieur.

CONTENTS

	INTRODUCTION	1
I	SOME DEFINITIONS AND PRELIMINARY LEMMAS	11
II	THE N-PARTICLE IRREDUCIBLE KERNELS S^{N+1} AND THE BETHE-SALPETER KERNEL	30
III	DERIVATIVES WITH RESPECT TO t	42
IV	ANALYTICITY	57
V	SOME PERTURBATION EXPANSIONS	63
VI	THE POLES OF R AND S	69
VII	THE CONNECTION BETWEEN THE MASS SPECTRUM AND THE POLES OF S AND R_{22}	83
VIII	THE BOUND STATE	93
IX	A $P(\phi)_2$ MODEL WITH STRONG EXTERNAL FIELD	100
	REFERENCES	111

INTRODUCTION

In this paper we study the two particle bound states in weakly coupled $\lambda P(\phi)_2$ boson quantum field theories

$$(1) \quad (\partial_t^2 - \partial_x^2 + m^2) \varphi(t, \underline{x}) = -\lambda : P'(\varphi) : (t, \underline{x})$$

$$[\varphi(t, \underline{x}), \dot{\varphi}(t, \underline{x})] = -i \delta(\underline{x} - \underline{y})$$

where P is a polynomial of the form

$$P(x) = \sum_{j=0}^{2n} c_j x^j, \quad c_{2n} > 0.$$

For the construction of such theories the passage to imaginary time fields and Euclidean fields as proposed first by Symanzik turned out to be extremely useful.

In the Euclidean framework a set of symmetric Schwinger functions $S^{(n)}((t_1, \underline{x}_1), \dots, (t_n, \underline{x}_n))$ is first constructed satisfying the Osterwalder-Schrader axioms [OSCH]. They define in a unique way a Wightman field theory such that

$$S^{(n)}((t_1, \underline{x}_1), \dots, (t_n, \underline{x}_n)) = W^{(n)}((i t_{\pi(1)}, \underline{x}_{\pi(1)}), \dots, (i t_{\pi(n)}, \underline{x}_{\pi(n)}))$$

where Π is the permutation for which

$$t_{\pi(1)} \leq t_{\pi(2)} \leq \dots \leq t_{\pi(n)}, \quad \text{and}$$

$$W^{(n)}((t_1, \underline{x}_1), \dots, (t_n, \underline{x}_n)) = (\Omega, \varphi(t_1, \underline{x}_1) \dots \varphi(t_n, \underline{x}_n) \Omega)$$

are the Wightman functions of the theory. The field $\phi(t, \underline{x})$ obtained by this procedure satisfies the field equation (1) [Sch].

In our case (writing $S^{(n)} = S_{\phi, \dots, \phi}$ and $x_i = (t_i, \underline{x}_i)$)

$$\begin{aligned} (2) \quad S_{\phi, \dots, \phi}^{(n)}(x_1, \dots, x_n) &= \left\langle \prod_{i=1}^n \phi^{x_i}(x_i) \right\rangle \\ &= \lim_{h \rightarrow 1} \frac{\int d\mu_{w_0} \prod_{i=1}^n \phi^{x_i}(x_i) e^{-\lambda \int d^2x h(x) : \phi(\phi) : (x)}}{\int d\mu_{w_0} e^{-\lambda \int d^2x h(x) : \phi(\phi) : (x)}} \end{aligned}$$

where $d\mu_{w_0}^2$ denotes the Gaussian measure on $S^1(\mathbb{R}^2)$ with mean zero and covariance $C(x, y) = (-\Delta + m_0^2)^{-1}(x, y)$. It is known among other things that

([GJS], [D], [EEF], [DE], [OS])

- The so constructed functions satisfy the Osterwalder-Schrader axioms.

- The Schwinger functions are analytic in λ for λ in the region $|\lambda m_0^{-2}| < \epsilon$, $\text{Re} \lambda > 0$ and perturbation theory is asymptotic
- There is a unique vacuum and a mass gap $m(\lambda) > 0$
- The mass shell $p^2 = m^2(\lambda)$ is isolated, $|m(\lambda) - m_0| = O(\lambda)$, with no other spectrum up to $2m_0 - O(\lambda)$
- The physical mass $m(\lambda)$ (a pole of the two point function) and the field strength (its residuum) are C^∞ in λ for small $\lambda > 0$
- The S-matrix is non trivial

After these general results the efforts in $\lambda P(\phi)_2$ were concentrated on the study of the mass spectrum up to $3m_0 - O(\lambda)$. For even theories two results in this direction are

- Below $2m(\lambda)$ the mass spectrum is discrete and of finite multiplicity [SZ]
- If the coefficient C_4 of ϕ^4 in the interaction polynomial is positive then there is no mass spectrum in the (open) interval $(m(\lambda), 2m(\lambda))$ [SZ] and in the case $C_4 < 0$ there is exactly one two particle bound state and its mass $m_B(\lambda) < 2m(\lambda)$ is C^∞ in small $\lambda > 0$ [DE].

The purpose of this paper is to extend these results to general $P(\phi)_2$ models i.e. models including also odd powers in the interaction polynomial. Let $S(\lambda; k)$, $R_{21}(\lambda; k, p)$ and $R(\lambda; k, p, q)$ denote the Fourier transform of the truncated two

point function, the truncated three point function and the one particle irreducible four point function respectively. The criteria for the occurrence of two particle bound states can be written in terms of a kernel L and the Bethe-Salpeter kernel K :

$$(3) \quad L(\lambda; k, p) = [R(\lambda; k)^{-1} R_{21}(\lambda; k, \cdot)](p) \cdot S(\lambda; k)^{-1}$$

$$(4) \quad K(\lambda; k, p, q) = R(\lambda; k)^{-1}(p, q) - R_0(\lambda; k)^{-1}(p, q)$$

$$\text{where} \quad R_0(\lambda; k, p, q) = 4\pi \, S(\lambda; \frac{k}{2} + p) S(\lambda; \frac{k}{2} - p) \delta(p+q) \quad .$$

More precisely let

$$a) \quad K(\lambda; k, p, q) = \mathcal{O}(\lambda^n)$$

$$\alpha_n := (n!)^{-1} \partial_\lambda^n K(0; (i \cdot 2\omega_0, 0), 0, 0) \neq 0$$

$$b) \quad L(\lambda; k, p) = \mathcal{O}(\lambda^m)$$

$$\beta_m := (m!)^{-1} \partial_\lambda^m L(0; (i \cdot 2\omega_0, 0), 0) \neq 0$$

$$c) \quad \gamma := -\frac{1}{6\pi} \omega_0^{-2} \beta_m^2 \alpha_n^{-1} \neq 1$$

$$\text{Then with} \quad \hat{\psi}(x) = \psi(x) - (\psi(x), \Omega, \Omega)$$

$$(5) \quad (\hat{\psi}(x)\mathcal{Q}, \hat{\psi}(y)\mathcal{Q}) = \mathcal{Z}(\lambda) \Delta_+(\langle x-y \rangle, w^2(\lambda)) \\ + \mathcal{Z}_2(\lambda) \Delta_+(\langle x-y \rangle, w_B^2(\lambda)) + \int_{4w^2(\lambda)} d\mu_\lambda(a) \Delta_+(\langle x-y \rangle, a)$$

$$(6) \quad (:\hat{\psi}^2(x)\mathcal{Q}, :\hat{\psi}^2(y)\mathcal{Q}) = \mathcal{Z}_3(\lambda) \Delta_+(\langle x-y \rangle, w^2(\lambda)) \\ + \mathcal{Z}_4(\lambda) \Delta_+(\langle x-y \rangle, w_B^2(\lambda)) + \int_{4w^2(\lambda)} d\mathcal{S}_\lambda(a) \Delta_+(\langle x-y \rangle, a)$$

and we can prove the following

THEOREM

For $\lambda \geq 0$ sufficiently small we have

1) If $\alpha_n > 0$ or $\alpha_n < 0$ with $n = 2m$ and $\gamma > 1$ then there is
no two particle bound state.

2) In the remaining cases there is exactly one two particle
bound state for $\lambda > 0$. Its mass $m_B(\lambda)$ is C^∞ in λ with

$$w_B^2(\lambda) = 4w^2(\lambda) - (1 - \lambda^{2m+\gamma} \delta)^2 w_0^2 (\pi w_0^{2(m-1)} \alpha_n)^2 \left(\frac{\lambda}{w_0^2}\right)^{2m} \\ + O\left(\left(\frac{\lambda}{w_0^2}\right)^{2m+1}\right) .$$

For $L \equiv 0$ $\mathcal{Z}_1(\lambda)$ is C^∞ in λ and

$$\mathcal{Z}_1(\lambda) = -(\pi w_0^{2(m-1)} \alpha_n) \left(\frac{\lambda}{w_0^2}\right)^m + O\left(\left(\frac{\lambda}{w_0^2}\right)^{m+1}\right) .$$

If $L \neq 0$ $\mathcal{Z}_2(\lambda)$ is C^∞ in λ and

$$\mathcal{Z}_2(\lambda) = -\frac{1}{9} (1 - \lambda^{2m+\gamma} \delta) (\pi w_0^{2(m-1)} \alpha_n) (w_0^{2(m-1)} \beta_m)^2 \left(\frac{\lambda}{w_0^2}\right)^{2m+m} \\ + O\left(\left(\frac{\lambda}{w_0^2}\right)^{2m+m+1}\right) .$$

Our analysis can also be applied to $P(\lambda)_2$ - models with

strong external field. (More details are given in Theorem 34 p. 106).

THEOREM

In the $\lambda\phi^4 + 4\mu\phi$ theory with $\lambda > 0$ and large $|\mu|$ there are exactly two particles with mass less than $2\bar{m}(\mu)$ where

$$\bar{m}(\mu)^2 = 12\lambda^{1/3}\mu^{2/3} + O(\log|\mu|) .$$

Their masses are C^∞ in small $\mu^{-1/3} \neq 0$.

We shall give a short sketch of the reasons why bound states occur. However this can not be done without using some results of the Sections IV and VI such as the fact that particles with mass $< 2m(\lambda)$ show up as poles in the analytic continuation $S(\lambda; \chi)$ of the Euclidean two point function $S(\lambda; k)$ to momenta $k = (i\chi, 0)$, or in the analytic continuation $R_{22}(\lambda; \chi, p, q)$ of the four point function $R_{22}(\lambda, k, p, q)$ ($= R(\lambda; k, p, q)$ in even theories).

Consider the equation

$$(7) \quad S(\lambda; \chi) = \zeta(\chi) (1 + 2\bar{u} k(\lambda; \chi) \zeta(\chi))^{-1}$$

for $S(\lambda; \chi)$. $\zeta(\chi) = (2\bar{u})^{-1} (-\chi^2 + u_0^2)^{-1}$.

Since the one particle irreducible two point function

$k(\lambda; \chi)$ is of order $O(\lambda)$ and analytic for

$|\operatorname{Re} \chi| < 2m_0 - O(\lambda)$ (see C.23) we observe the well known fact that $S(\lambda; \chi)$ has a pole i.e.

$$(8) \quad 2\pi k(\lambda; \chi) C(\chi) = -1$$

for some value $m(\lambda)$ of χ near m_0 .

An other pole which is relevant for the mass spectrum below $2m(\lambda)$ is that of

$$(9) \quad R_{22}(\lambda; \chi, p, q) = R(\lambda; \chi, p, q) + R_{21}(\lambda; \chi, p) S(\lambda; \chi)^{-1} R_{12}(\lambda; \chi, q)$$

The first term can be studied by looking at the operator equation (see (4))

$$(10) \quad R(\lambda; \chi) = R_0(\lambda; \chi) (1 + K(\lambda; \chi) R_0(\lambda; \chi))^{-1}$$

Since the Bethe-Salpeter kernel $K(\lambda; \chi, p, q)$ is of order $O(\lambda)$ (see C.23) and since $R_0(\lambda; \chi)$ has (in two space-time dimensions) a kinematical singularity $R_0 \sim (4m^2(\lambda) - \chi^2)^{-\frac{1}{2}}$ we conclude by (10) that $R(\lambda; \chi)$ can have a pole at $\chi = \chi_1(\lambda)$ near (and below) $2m(\lambda)$ where $f + K(\lambda; \chi_1(\lambda)) R_0(\lambda; \chi_1(\lambda)) f = 0$ for some function f . In an even theory this leads to a bound state with mass $\chi_1(\lambda)$

(see also [DE]).

Before considering R_{22} in the odd case let us first return to (7) and look at the other possibility for $S(\lambda; \chi)$ to have a pole below $2m(\lambda)$, namely when (8) holds because $k(\lambda; \chi)$ has a pole. Since

$$(11) \quad k(\lambda; \chi) = -(2\pi)^{-1} \langle L(\lambda; \chi, \cdot), R(\lambda; \chi) L(\lambda; \chi, \cdot) \rangle_{\mathcal{L}^2} + \text{regular}$$

with the one particle irreducible three point function L defined in (3) which is $O(\lambda)$ and regular (see C.23) we see that in an even theory where three point functions vanish $k(\lambda; \chi)$ is also regular. Regular here means analytic in $|\text{Re} \chi| < 3m_0 - O(\lambda)$. In this case $S(\lambda; \chi)$ has no additional pole below $2m(\lambda)$.

But in an odd theory the situation is different. L does not vanish and thus by (11) $k(\lambda; \chi)$ can have a pole at $\chi = \chi_1(\lambda)$ induced by $R(\lambda; \chi)$. Then (8) may hold for some value $\chi_2(\lambda)$ of χ near $\chi_1(\lambda)$ (and below $2m(\lambda)$) leading to a pole in $S(\lambda; \chi)$ i.e. a two particle bound state with mass $\chi_2(\lambda)$.

Note that in this case $S(\lambda; \chi)$ has a C.D.D. - zero at $\chi = \chi_1(\lambda)$. Thus for odd theories both terms on the right hand side of (9) may have a pole at $\chi = \chi_1(\lambda)$. But by using the fact that the singularity of $R(\lambda; \chi)$ at $\chi_1(\lambda)$ is contained in a rank

one operator it can be shown (see the proof of p.) that the poles of $R(\lambda; \chi)$ and $R_{21}(\lambda; \chi) S(\lambda; \chi)^{-1} R_{21}(\lambda; \chi)^*$ always cancel. Consequently $R_{22}(\lambda; \chi)$ is analytic at $\chi_1(\lambda)$ (but has a pole at $\chi_2(\lambda)$) if $L \neq 0$.

A large part of this paper is devoted to the proof of analyticity properties of kernels like $k(\lambda; \chi)$, $L(\lambda; \chi)$, $K(\lambda; \chi)$ etc. The method which we use is due to Spencer [S II]. We shall illustrate it for the case of the n -particle irreducible expectations $\langle Q_1; Q_2 \rangle^{\geq n+1}$ which play an essential role in the many-particle structure analysis initiated by Symanzik [SY]; see also [B], [BL], [CO].

We define $\langle Q_1(X); Q_2(Y) \rangle^{\geq n}$ first formally as the sum of all graphs of $\langle Q_1(X) Q_2(Y) \rangle$ with at least n lines hitting each line $\ell \subset \mathbb{R}^2$ separating X from Y . An example for such a graph is

$$(12) \quad \prod_{i=1}^n C(x_i, y_i) \quad .$$

Since $C(x, y) = O(e^{-m_0 |x - y|})$ the kernels $\langle Q_1(X); Q_2(Y) \rangle^{\geq n}$ should decay as $e^{-n(m_0 - \varepsilon) \text{dist}(x, y)}$ or equivalently their Fourier-Laplace transform

$$(2\pi)^{-1} \int d\tau e^{i\tau k} \langle Q_1(x); Q_2(\{y_1 + \tau, y_2 + \tau, \dots\}) \rangle^{\geq n}$$

should for real k^1 be analytic in $|\operatorname{Im} k^0| < nm_0(1 - \epsilon)$. To prove this by starting with another definition of $\langle Q_1; Q_2 \rangle^{\geq n}$ (see Section II) we show first that it is in some sense equivalent to the first one. This can be done as follows. Let Δ_ℓ denote the Laplacian with zero boundary conditions on a (straight) line $\ell \subset \mathbb{R}^2$. We define $C_\ell = (-\Delta_\ell + m_0^2)^{-1}$,

$$C(t, x, y) = t C(x, y) + (1-t) C_\ell(x, y)$$

and $\langle \dots \rangle_t$ as the expectation with respect to the Gaussian measure $d\mu_{m_0^2}(t)$ with covariance $C(t, x, y)$. Notice that $C_\ell(x, y) = 0$ if ℓ separates x from y . Consequently

$$\prod_{i=1}^n C(t, x_i, y_i) = O(t^n)$$

If ℓ separates $\{X_i\}$ from $\{Y_i\}$. In fact it suffices to show that

$$\partial_t^k \langle Q_1(x); Q_2(y) \rangle_t^{\geq n} \Big|_{t=0} = 0 \quad k=0, 1, \dots, n-1$$

independent of the line ℓ (parallel to $x^1 = 0$) in order to obtain an exponential decay $e^{-nm_0(1 - \epsilon)\min|y_i^0 - x_j^0|}$ and the desired analyticity in p-space.

Such t -derivatives are computed in Section III for the different kernels defined in Section II. The resulting analyticity properties (see Sections IV, VI) are related to the mass spectrum as will be outlined in Section VII. Problems concerning the existence of bound states, their masses etc. are treated in Section VIII by using C^∞ properties in λ established in V. The last section is devoted to the application of these results to a $P(\phi)_2$ theory with strong external field.

Remark : Recently Glimm and Jaffe [GJII] also proved that the mass spectrum in weakly coupled $\lambda P(\phi)_2$ below the $2m$ threshold is isolated. They use physical one particle subtractions combined with an expansion as in [GJS I].

I SOME DEFINITIONS AND PRELIMINARY LEMMAS

In this section we introduce classes $A_{p,q}^{\sigma,\tau}$ of bounded linear maps such that each kernel introduced later defines an element in some $A_{p,q}^{\sigma,\tau}$. Sums, products, tensor products and some inverses will be defined.

But first we illustrate how the decay property of a kernel K_t constructed with t -expectations $\langle \dots \rangle_t$ is obtained by using the fact that certain of its t -derivatives vanish at $t = 0$. To do this we need simultaneous derivatives at different lines

$$\mathcal{L}_i = \{ (x^0, x^1) \in \mathbb{R}^2 : x^0 = i \}$$

for $i \in I$ and $I \subset \mathbb{Z}$ to be specified later (depending on K_t , see Section IV). Let $\alpha : I \rightarrow \mathbb{N}$ and $t : I \rightarrow [0,1]$ be two functions on I , sometimes written as $\alpha = \{\alpha(i)\}_{i \in I}$, $T = \{t_i\}_{i \in I}$. Then $d\mu_{m_0}^2(t)$ and $\langle \dots \rangle_t$ are defined with respect to the new covariance

$$(I.1) \quad C(t, x, y) = \prod_{i \in I} (t_i + (1-t_i) J_{\epsilon_i}) C_\phi(x, y)$$

where $J_{\epsilon'} C_\epsilon = C_{\epsilon' \vee \epsilon}$. Suppose that we have shown

$$\partial_t^\alpha K_t := \prod_{i \in I} \frac{\partial^{\alpha(i)}}{\partial t^{\alpha(i)}} K_t = 0 \quad \text{at} \quad t=0$$

for every function $\alpha < \beta$ (i.e. $\alpha(i) < \beta(i) \forall i \in I$). Then

$$(I.2) \quad K = K_1 = \int_{t=0}^1 dt \prod_{i \in I} \frac{(1-t_i)^{\beta(i)-1}}{(\beta(i)-1)!} \partial_t^\beta K_t.$$

The t_i -derivatives are computed by using [DG] :

$$\partial_{t_i} \int Q e^{-\lambda V} d\mu(t) = \int [\partial_{t_i} C \cdot \Delta_\phi] Q e^{-\lambda V} d\mu(t)$$

$$\text{with} \quad \partial_{t_i} C \cdot \Delta_\phi = \frac{1}{2} \int d^2x d^2y \partial_{t_i} C(t, x, y) \frac{\delta^2}{\delta \phi(x) \delta \phi(y)}.$$

For multiple derivatives the Leibnitz rule leads to

$$(I.3) \quad \partial_t^\alpha \int Q e^{-\lambda V} d\mu(t) = \sum_{\substack{\alpha_1 + \dots + \alpha_j = \alpha \\ \alpha_j \leq 1}} \int \frac{1}{j!} [\partial_t^{\alpha_j} C \cdot \Delta \phi] Q e^{-\lambda V} d\mu(t).$$

Next we can use that $d_t^\alpha C(t, x, y) = O(e^{-m_0 d(\alpha)})$ with

$$d(\alpha) = \begin{cases} \max \{ |i-j| : \alpha(i), \alpha(j) \neq 0 \} & \text{if } \alpha \leq 1 \\ \infty & \text{otherwise (i.e. if any } \alpha(i) > 1) \end{cases}$$

such that $d_t^\alpha C(t, x, y)$ could be replaced by

$$h(\alpha) \partial_t^\alpha C(t, x, y) \text{ for } h(\alpha) \leq e^{m_0(1-\varepsilon)d(\alpha)}.$$

This is the idea for defining the following modified expectations [SII] :

$$\langle Q \rangle_{t,h} = \frac{\int \prod_{\alpha \in \chi_1} \prod_{i \in \mathbb{Z}^2} (1+h(\alpha) \partial_t^\alpha C_{(i)} \Delta \phi) Q e^{-\lambda V} d\mu(t)}{\int \prod_{\alpha \in \chi_1} \prod_{i \in \mathbb{Z}^2} (1+h(\alpha) \partial_t^\alpha C_{(i)} \Delta \phi) e^{-\lambda V} d\mu(t)}.$$

$C_{(i)}(t, x, y) = \chi_{\Delta_{j_i}}(x) C(t, x, y) \chi_{\Delta_{j_i}}$ and $\chi_{\Delta_{j_i}}$ is the characteristic function of the unit square

$$\Delta_j = [j^0, j^0+1] \times [j^1, j^1+1] \quad \text{for } j = (j^0, j^1) \in \mathbb{Z}^2.$$

With the definition

$$\partial_t^\alpha \int Q e^{-\lambda V} d\mu(t) = \sum_{\substack{\alpha_1 + \dots + \alpha_j = \alpha \\ \alpha_j \leq 1}} \int \frac{1}{j!} \partial_{h(\alpha_j)} \langle Q \rangle_{t,h} \Big|_{h=0}$$

and an analogous formula is valid for kernels $K_{t,h} =$

$\sum_{i,j} \pi \langle Q_{ij} \rangle_{th}$. Combined with (I.2) this leads to the basic expansion

$$(I.4) \quad K = \int_{\{0\}}^{\{1\}} dt \prod_{i \in I} \frac{(1-t_i)^{\beta(i)-1}}{(\beta(i)-1)!} \sum_{\substack{\alpha_1 + \dots + \alpha_L = \beta \\ \alpha_j \geq 1}} \oint_{\partial R} dh K_{t,h} \prod_{j=1}^L \frac{h(\alpha_j)^{-2}}{2u^2}$$

which is true under the condition that

$$\partial_t^\alpha K_t \Big|_{t=0} = 0 \quad \text{for every} \quad \alpha \in \beta$$

and that $K_{t,h}$ is analytic in

$$R = \{h = h(\alpha) : |h(\alpha)| \leq \exp(u_0(1-\varepsilon)(d(\alpha)+1))\}.$$

It is the factor $\prod_{j=1}^L h(\alpha_j)^{-2}$ which will finally give the desired exponential decay. But we need also uniform bounds on kernels $K_{t,h}$ for $I \in \mathbb{Z}$ and $h \in R$.

The main input in this direction is the Theorem 3 in [SII] with a slight generalization which is also contained in the proof given there.

LEMMA 1 : For given $\varepsilon > 0$ and $1 < p \leq \infty$ there are positive

constants C_1, C_2, C_3 such that for

$0 \leq \lambda \leq C_1$, $u_0 \geq C_2$, $G_1, \dots, G_m \subset \mathbb{R}^2$ and $\omega \in L_p(\bigotimes_{i=1}^m G_i^{\otimes n_i})$
with support in a product of unit squares the integral

$$\int \frac{\omega}{h} d^{2u_i} x^i \omega(x) \left\langle \frac{n_1}{h} : \phi^{q_{i_1}}(x_{j_1}^1); \dots; \frac{n_m}{h} : \phi^{q_{i_m}}(x_{j_m}^m) \right\rangle_{t, h}$$

is bounded by $C_3 \exp \left[- \sum_{i < j} \text{dist}(G_i, G_j) \right] \cdot \| \omega \|_{L_p} < \infty$ uniformly
in $I \subset \mathbb{Z}, h \in \mathbb{R}$ and is analytic in h for $h \in \mathbb{R}$

$$R = \{ h = h(\alpha) : |h(\alpha)| \leq \exp(u_0(1-\varepsilon)(d(\alpha)+1)) \}$$

For the "covariance" $C(t, h, x, y) = \langle \phi(x); \phi(y) \rangle_{t, h} \Big|_{\lambda=0}$
 we need an additional property, namely

LEMMA 2 : For m_0 sufficiently large (depending on ε)

$$\sup_y \| C_q(t, h, \cdot, y) \|_{L_p} \leq C(p) e^{-|j_1 - j_2|} < \infty$$

If $1 \leq p < \infty$ and $h \in \mathbb{R} = \{ h = h(\alpha) : |h(\alpha)| \leq \exp(u_0(1-\varepsilon)(d(\alpha)+1)) \}$.

Proof :

Two standard estimates (see [GJS], [S], ...) based on the Wiener path representation can in our case be written as

$$(I.5) \quad 0 \leq \partial_t^\alpha C(t, x, y) \leq C(x, y) \leq c_1(\delta)(1 + \log |x - y|) e^{-u_0(1-\delta)|x-y|}$$

$$(I.6) \quad \partial_t^\alpha C(t, x, y) \leq c_2(\delta) e^{-u_0(1-\delta)d(\alpha)} \sup_{z \in L(\alpha)} C(z, x)$$

if $\alpha \neq 0$

where $L(\alpha) = \{l_i : \alpha(i) = 1\}$, and $\delta > 0$ can be chosen arbitrarily small.

Let $d(\alpha, j) = \min \{|i-j| : \alpha(i) = 1\}$. Then

$$\sup_{z \in L(\alpha)} C(z, x) \chi_{\Delta_{j_1}}(x) \leq C_3 \|1 + \log |x - j_1^0|\| e^{-w_0(1-\delta)d(\alpha, j_1)}$$

by (I.5). By combining this with (I.5), (I.6) we can bound derivatives of $C_{(j)}(t, x, y)$ as follows :

$$\begin{aligned} \text{(I.7)} \quad \partial_t^\alpha C_{(j)}(t, x, y) &= (\partial_t^\alpha C_{(j)}(t, x, y))^\delta (\partial_t^\alpha C_{(j)}(t, x, y))^{1-\delta} \\ &\leq C_4 (1 + \log |x - y|)^\delta (1 + \log |x - j_1^0|)^{1-\delta} e^{-w_0(1-2\delta)d(\alpha)} \\ &\quad \times e^{-w_0(1-2\delta)d(\alpha, j_1)} e^{-w_0 \frac{\delta}{2} |j_1 - j_2|} . \end{aligned}$$

Next we will use this estimate to bound $C_{(j)}(t, h, x, y)$ which is by definition equal to

$$\text{(I.8)} \quad C_{(j)}(t, h, x, y) = \sum_{\alpha: L(\alpha) \subset I} h(\alpha) \partial_t^\alpha C_{(j)}(t, x, y) .$$

Note that the number of functions in

$A_{d,r} = \{\alpha : \alpha \leq \chi_I, d(\alpha) = d \text{ and } d(\alpha, j_1) = r\}$ is

bounded by 2^{d+1} . Thus for $h \in R$

$$(I.9) \quad |C_{ij}(t, h, x, y)| \leq \sum_{d=0}^{\infty} \sum_{r=0}^{\infty} \sum_{\alpha \in A_{d,r}} e^{w_0(1-\varepsilon)(d+r)} |d_t^\alpha C_{ij}(t, x, y)| \\ \leq \sum_{d,r=0}^{\infty} 2^{d+1} e^{w_0(1-\varepsilon)(d+r)} \sup_{\alpha \in A_{d,r}} |C_{ij}(t, x, y)|.$$

By choosing $\delta = \varepsilon/4$ this sum is for sufficiently large m_0 controlled by the factor $e^{-m_0(1-2\delta)(d+r)}$ from (I.7) and we obtain

$$(I.10) \quad |C_{ij}(t, h, x, y)| \leq c_s (1 + |\log |x-y||) (1 + |\log |x-j_0||) e^{-w_0 \frac{\delta}{2} |j_0-j_1|}.$$

The assertion follows now immediately for $m_0 > 4\delta^{-1}$.

□

By their local L_p - and exponential decay properties some of the functions described in Lemma 1 and Lemma 2 represent kernels of bounded linear maps between Banach spaces $L_{p,q}^{n,\sigma}$ whose definition will be prepared now.

Definition 1 :

a) Let $L_{p,q}^m$ denote the subspace of $L_q^{loc}(\mathbb{R}^{2m})$ containing symmetric functions f for which

$$\|f\|_{p,q}^{(\omega)} = \|P_{A_j} f\|_{L_q(\mathbb{R}^{2m})} \|e_p(\mathbb{R}^{2m})\| \quad \text{is finite and}$$

$\|P_{\Delta_j} f\|_{L_q} \rightarrow 0$ as $|j| \rightarrow \infty$. Here P_{Δ_j} denotes the projection onto the functions with support in

$\Delta_j = \Delta_{j_1} \times \dots \times \Delta_{j_m}$. Notice that

$$\|f\|_{k, \ell}^{(\omega)} \leq \|f\|_{k, q}^{(\omega)} \leq \|f\|_{p, q}^{(\omega)} \quad \text{if } \ell \leq q, p \leq k.$$

b) Let $d(S)$ be the length of the shortest tree in $\mathbb{R}^2$ connecting every point of $S \subset \mathbb{R}^2$

Example : $S = \{j_1, j_2, j_3\}$

shortest tree



Then for $i \in \mathbb{Z}^{2m}, j \in \mathbb{Z}^{2n}$ we define

$$(I.11) \quad d(i, j) = \min_{k, \alpha_1, \alpha_2} \sum_{\ell=1}^k d(\alpha_1^\ell \cup \alpha_2^\ell)$$

where $\alpha_1 = \{\alpha_1^1, \dots, \alpha_1^k\}$ and $\alpha_2 = \{\alpha_2^1, \dots, \alpha_2^k\}$ denote partitions of $\{i_1, \dots, i_m\}$ and $\{j_1, \dots, j_n\}$ respectively.

Among the bounded linear maps $A : L_{\infty, q}^n \rightarrow L_{\infty, q}^n$ (for some p, q) we are interested in those having the following exponential

decay property :

(I.12) There is a constant $c = c(A) > 0$ such that

$$|A|_{p,q}^{(c)} := \sup_{i,j} \|P_{A_i} A P_{A_j}\|_{p,q} e^{cd(i,j)} \quad \text{is finite}$$

where $\|\cdot\|_{p,q}$ denotes the norm of a continuous map from L_q^n to L_p^m , $L_p^m = L_{\infty,p}^m$.

Proposition 3 : Let $A_1, A_2 : L_q^m \rightarrow L_p^k$
and $A_3 : L_r^n \rightarrow L_q^m$ be given
satisfying condition (I.12) for $c(A_i) = c_i$. Then there are
positive constants k_1, k_2, k_3 only depending on k, m, n, c_1, c_2, c_3
such that

$$a) \quad |A_1 + A_2|_{p,q}^{(\min(c_1, c_2))} \leq \|A_1\|_{p,q}^{(c_1)} + \|A_2\|_{p,q}^{(c_2)}.$$

$$b) \quad |A_2 A_3|_{p,r}^{(k_1)} \leq k_2 |A_2|_{p,q}^{(c_2)} \cdot |A_3|_{p,q}^{(c_3)}$$

$$c) \quad \|A_1\|_{p,q} \leq k_3 |A_1|_{p,q}^{(c_1)}$$

Proof :

a) is obvious.

b) is proved as (I.19) below.

c) Let B denote the unit ball in L_q^m . Then

$$\begin{aligned} \|A_1\|_{p,q} &= \sup_{f \in B} \sup_i \|P_{\Delta_i} A_1 f\|_{L_p} \leq \sup_{f \in B} \sup_i \sum_j \|P_{\Delta_i} A_1 P_{\Delta_j} f\|_{L_p} \\ &\leq \sup_i \sum_j \|P_{\Delta_i} A_1 P_{\Delta_j}\|_{p,q} \leq \sup_i \sum_j e^{-c_1 d(i,j)} |A_1|_{p,q}^{(c_1)} \leq k_3 |A_1|_{p,q}^{(c_1)} \end{aligned}$$

and obviously $A_1 L_q^m c L_p^k$.

□

The objects from which we construct our kernels are partially amputated Schwinger functions defined as follows (we omit the index t, h)

Definition 2 :

Let $K = \{1, \dots, m\}$, $L = \{m+1, \dots, m+n\}$. Then

$$\begin{aligned} S(x_1, \dots, x_m; x_{m+1}, \dots, x_{m+n}) &= \delta_{m,0} \delta_{n,2} \mathcal{L}(x_1 - x_2) + \delta_{m,2} \delta_{n,1} \mathcal{L}(x_1 - x_2) \\ &+ \sum_{\Pi} (-1)^{|\Pi|} \int \prod_{i \in \Pi \cap K} d\gamma_i \prod_{i \in \Pi \cap L} \delta(x_i - \gamma_i) \prod_{i \in \Pi \cap L} \mathcal{L}(x_i - \gamma_i) \langle \prod_{i \in \Pi} \mathcal{P}^{(151)}(\emptyset) : (\gamma_i) \rangle^T \end{aligned}$$

where the sum runs over the partitions Π of $\{1, \dots, m+n\}$ and $\mathcal{P}^{(k)}$ denotes the k -th derivative of the interaction polynomial P . The corresponding untruncated functions $S(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+n})$ and partially truncated functions are constructed as usually (see also (III.5)) from the above defined kernels $S(x_1, \dots, x_m; x_{m+1}, \dots, x_{m+n})$.

Remark : The so defined (untruncated) functions do (for $m > 1$) not coincide with what is usually called amputated Schwinger functions, since

$$\int d^2x' d^2y' C(x-x') S(x', y') C(y'-y) = S(x, y) - C(x, y) .$$

For a recursion formula $m \rightarrow m + 1$ see (III.4).

If an integral as $\int d^{2n} \gamma S(x_1, \dots, x_m; y_1, \dots, y_n) f(\gamma)$ should make sense f must at least be restrictable to hyperplanes

$$H_{\pi}^n = \{(\gamma_1, \dots, \gamma_n) \in \mathbb{R}^{2n} : \gamma_i = \gamma_j \text{ if } i, j \in \pi^k \text{ for some } \pi^k \in \pi\}$$

defined by partitions $\pi = \{\pi^1, \dots, \pi^{|\pi|}\}$ of $\{1, \dots, n\}$. Spaces

$L_{p,q}(H_{\pi}^n)$ can be introduced as in Definition 1 and since $\dim$

$H_{\pi}^n = 2|\pi|$ we may denote the norm in $L_{p,q}(H_{\pi}^n)$ also by

$\|\cdot\|_{p,q}^{(|\pi|)}$. So Definition 2 motivates the introduction of

$$\bigoplus_{\pi \in \mathcal{P}(1, \dots, n)} L_{p,q}(H_{\pi}^n) .$$

To preserve the notation of functions we do this in the following two equivalent ways:

Definition 3

Let $L_{p,q}^n$ denote the completion of $C_0^{\text{symm.}}(\mathbb{R}^{2n})$ with respect

to the norm

$$\| \omega \|_{p,q}^{(u)} = \left(\sum_{\pi \in P(1, \dots, n)} \| \omega_{\pi} \|_{p,q}^{(1\pi)}^2 \right)^{1/2}$$

where $w_{\Pi} = w \upharpoonright H_{\Pi}^n$. By $L_{q,p}^{n,*}$ we denote the Banach space of symmetric functions

$$f(x_1, \dots, x_n) = \sum_{\pi \in P(1, \dots, n)} \int d^{2|\pi|} y \prod_{j=1}^{|\pi|} \overline{\pi} \delta(x_i - y_j) f_{\pi}(y_1, \dots, y_{|\pi|})$$

with $f_{\Pi} \in L_{p,q}(H_{\Pi}^n)$ and the norm

$$\| f \|_{p,q}^{(u)*} = \left(\sum_{\pi \in P(1, \dots, n)} \| f_{\pi} \|_{p,q}^{(1\pi)}^2 \right)^{1/2}.$$

A product on $L_{p^*,q^*}^{n,*} \times L_{p,q}^n$ (p^*, q^* are the dual Hölder indices of p, q) can be defined by

$$(I.13) \quad \langle \omega, f \rangle_n = \int d^{2n} x f(x_1, \dots, x_n) \overline{\omega(x_1, \dots, x_n)} = \sum_{\pi \in P(1, \dots, n)} \langle f_{\pi}, \omega_{\pi} \rangle_{L_2(H_{\pi}^n)}.$$

Before defining our main class of operators let us introduce

variables $\sigma, \tau, \dots$ with values * or "no*" and let $L_q^{n\sigma} = L_{\infty,q}^{n,\sigma}$.

Definition 4

Linear maps $A : L_q^{n\tau} \rightarrow L_p^{m\sigma}$ are said to be in $A_{p,q}^{\sigma,\tau}$ if their components $A_{\Pi_1, \Pi_2} : L_{\infty, q}^{(H_{\Pi_2}^n)} \rightarrow L_{\infty, p}^{(H_{\Pi_1}^m)}$ defined by

$$(A\varphi)_{\pi_1} = \sum_{\pi_2 \in \mathcal{O}(i_1, \dots, i_n)} A_{\pi_1, \pi_2} \varphi_{\pi_2}$$

satisfy the "tree decay" condition (I.12).

To simplify the notations we omit injections which indentify $A_{r,s}^{\sigma,\tau}$ with a subset of $A_{p,q}^{\sigma,\tau}$ if $p \leq r, s \leq q$.

Furthermore let $\phi \rightarrow \int dY K(.; Y) \phi(Y)$ denote the map defined by the kernel $K(x; Y)$.

Proposition 4 : For given $1 < p < q < \infty$ and sufficiently small
 λ and m_0^{-2} (depending on p, q)

a) The kernel $C_n(x; Y) = n! \prod_{j=1}^n C(t, h, x_j, Y_j)$ defines an
element $C_n \in A_{p,1}^{*,*}$.

b) The kernel

$A(x; Y) = S(x_1; \dots; x_m; x_{m+1}; \dots; x_k; Y_1; \dots; Y_{n+1}; \dots; Y_e)$
defines an element

$A \in A_{p,q}^{*,*}$ if $m = k, n = 1$

$A \in A_{p,1}^{*,*}$ if $m = k, n = 0, (k, e) \neq (1, 1)$

$A \in A_{p,1}^{*,*}$ if $m = 0, n = 0$.

Proof :

- a) $C_n(t, h)_{\Pi_1, \Pi_2}(x_1, \dots, x_{|\Pi_1|}, y_1, \dots, y_{|\Pi_2|})$ is up to permutations of $\underline{x}$ and $\underline{y}$ equal to

$$n! \prod_{i=1}^n C(t, h, x'_i, y'_i) \quad \left| \begin{array}{l} x'_i = x_j \quad \text{if } i \in \pi_1^j \\ y'_i = y_i \quad \text{if } i \in \pi_2^j \end{array} \right.$$

Thus by the definition of $|\cdot|_{p,1}^{(1)}$

$$\begin{aligned} |C_n(t, h)_{\pi_1, \pi_2}|_{p,1}^{(1)} &\leq \sup_{\Delta_i, \Delta_j} \sup_{\gamma \in \Delta_j} \|C_n(t, h)_{\pi_1, \pi_2}(\cdot, \gamma)\|_{L_p(\Delta_i)} e^{d(z,i)} \\ &\leq \left(\sup_{\gamma, j} \|C_{ij}(t, h, \cdot, \gamma)\|_{L_{np}} e^{(j_i - j_i^*)} \right)^n \cdot n! \quad (\text{Hölder}) \\ &\leq n! C(np)^n < \infty \quad (\text{Lemma 2}) \end{aligned}$$

- b) Let $k, \ell > 1$. The decay property (I.12) follows from Lemma 1 so that it suffices to have bounds of the form

$$(I, 14) \quad \|P_{\Delta_i} A_{\pi_1, \pi_2} P_{\Delta_j}\|_{p,q} < \infty$$

for arbitrary Δ_1, Δ_j . We consider first the case $k=m, \ell=n$.

By construction (Definition 2) the kernel of each component

A_{Π_1, Π_2} must be a finite sum of terms of the form

$$\left\langle \frac{1}{H} : \mathcal{P}^{(\dots)}(\phi) : (x_k) \frac{1}{H} : \mathcal{P}^{(\dots)}(\phi) : (y_k) \right\rangle^T \delta(x_1 - y_1) \dots \delta(x_\lambda - y_\lambda)$$

$k=1$ $k=2+1$

whose first factor which we denote $h_\lambda(x_1, \dots, y_k)$ is be Lemma 1 in L_r^{loc} for $1 < r < \infty$. Thus by choosing $r^{-1} = p^{-1} - q^{-1}$ and applying the Hölder inequality, we get

$$\int_{\Delta_i \times \Delta_j} d\underline{x} d\underline{y} d\underline{z} f(\underline{x}, \underline{z}) h_\lambda(\underline{x}, \underline{z}, \underline{y}) g(\underline{z}, \underline{y}) /$$

$$\leq \|f\|_{L_p} \|g\|_{L_q} \|h_\lambda\|_{L_r(\Delta_i \times \Delta_j)}$$

and we can bound the left hand side of (I.14) by

$$\sum_{k=0}^{w_{i,n}(|T_1|, |T_2|)} \|h_\lambda\|_{L_r(\Delta_i \times \Delta_j)}.$$

The remaining cases follow by multiplying with $C(t, h)_k$ from the right or $C(t, h)_\ell$ from the left using Proposition 3 b).

The preceding definitions allow to introduce the tensor product $A_1 \otimes A_2$ for $A_1, A_2 \in 1 < p < \infty$ $A_{p,1}^{\sigma, \tau}$. Notice that by the theorem of Dunford-Pettis [T], if $A_{\Pi_1, \Pi_2} \in A_{p,1}^{\sigma, \tau}$ then A_{Π_1, Π_2} is an integral operator whose kernel $A_{\Pi_1, \Pi_2}(\underline{x}; \underline{y})$ satisfies

$$(I.15) \quad \sup_{\underline{y} \in \Delta_j} \|P_{\Delta_i} A_{\tau_1, \tau_2}(\cdot; \underline{y})\|_{L_p} = \|P_{\Delta_i} A_{\tau_1, \tau_2} P_{\Delta_j}\|_{p,1}$$

We may thus define

Definition 5 :

$$(A_1 \otimes A_2)(x_1, \dots, x_{m_1+m_2}; y_1, \dots, y_{n_1+n_2}) = \frac{1}{(m_1+m_2)!(n_1+n_2)!} \\ \times \sum_{P, Q} A_1(x_{P(1)}, \dots, x_{P(m_1)}; y_{Q(1)}, \dots, y_{Q(n_1)}) A_2(x_{P(m_1+1)}, \dots, x_{P(m_1+m_2)}; y_{Q(n_1+1)}, \dots, y_{Q(n_1+n_2)})$$

Proposition 5 :

If $A_1, A_2 \in A_{q,1}^{s,r}$ then $A_1 \otimes A_2 \in A_{p,1}^{s,r}$ where
 $1 < p < \omega$, and $q = p$ if $G = *$, $q = 2p$ otherwise.

Proof :

For $h = 1, 2$ let the components of $A_k : L_1^{n_k, \tau} \rightarrow L_p^{m_k, \sigma}$ have the decay property (I.12). Then since

$d(i_1 \cup i_2, j_1 \cup j_2) \leq d(i_1, j_1) + d(i_2, j_2)$ it suffices to show

$$(I.16) \quad \|P_{A_1}(A_1 \otimes A_2)P_{A_1}\|_{p,1} \leq \|P_{A_1}(A_1)_{\pi_{12}, \pi_{22}}P_{A_1}\|_{2p,1} \|P_{A_2}(A_2)_{\pi_{31}, \pi_{32}}P_{A_2}\|_{2p,1}.$$

From D5 it follows that up to permutations of $X = \{x_1, \dots, x_{|\Pi_1|}\}$ and $Y = \{y_1, \dots, y_{|\Pi_2|}\}$

$$(I.17) \quad (A_1 \otimes A_2)_{\pi_1, \pi_2}(X; Y) = (A_1)_{\pi_{11}, \pi_{12}}(X_1; Y_1) \cdot (A_2)_{\pi_{21}, \pi_{22}}(X_2; Y_2)$$

$$\text{where } \pi_{11} = \{ \pi_{11}^k : \pi_{11}^k = \pi_1^k \cap \{i_1, \dots, w_1\} \neq \emptyset \}, \quad X_1 = \{ X_k : \pi_{11}^k \in \pi_{11} \},$$

$$\pi_{12} = \{ \pi_{12}^k : \pi_{12}^k = \pi_1^k \cap \{w_1+1, \dots, w_1+w_2\} \neq \emptyset \}, \quad X_2 = \{ X_k : \pi_{12}^k \in \pi_{12} \},$$

and analogously for $\pi_{21}, Y_1, \pi_{22}, Y_2$. Notice that

$$\text{for } \sigma = * : \quad (A_1 \otimes A_2)_{\pi_1, \pi_2} = 0 \quad \text{unless} \quad \pi_{11} \cap \pi_{22} = \emptyset$$

$$\text{for } \sigma \neq * : \quad (A_1 \otimes A_2)_{\pi_1, \pi_2} = 0 \quad \text{unless} \quad \pi_{12} \cap \pi_{22} = \emptyset.$$

Thus we obtain (I.16) by applying (I.15) and the Hölder inequality if $\sigma \neq *$.

□

Let l_k denote the injection of $l_q^{k\sigma}$ into $l_p^{k\sigma}$ for $q \leq p$ and let $A \in A_{p,1}^{\sigma\sigma}$. Then we define $l_k \otimes l_\ell = l_{k+\ell}$, and $l_k \otimes A$ formally as in D5 with

$$\mathbb{1}_k(x_1, \dots, x_k; y_1, \dots, y_k) = \delta(x_1 - y_1) \cdots \delta(x_k - y_k).$$

Proposition 6 :

- a) $1_k \otimes A \in A_{p,p}^{**}$ if $A \in A_{p,1}^{**}$, $1 < p < \infty$.
- b) $(1_n + \lambda A)^{-1} \in A_{p,p}^{\sigma\sigma}$ for small $|\lambda|$ if $A \in A_{p,p}^{\sigma\sigma}$ (acting
on $L_p^{n\sigma}$), $1 < p < \infty$.

Proof :

- a) Since $1_k \in A_{p,p}^{\sigma\sigma}$ the exponential decay of $1_k \otimes A$ is obvious as in Proposition 5. There remains to show that

$$(I.18) \quad \| (1_k \otimes A)_{\pi_1, \pi_2} \|_{P_{\Delta_i}} \leq \text{const.} \cdot \| f \|_{L_p}$$

By looking at (I.17) the right hand side of (I.18) is (e.g. in the case of three variables and $k = 2$) equal to

$$\begin{aligned} & \left(\int_{\Delta_i} d^6 x \int_{\Delta_j} d^6 y \delta(x_1 - y_1) \delta(x_2 - y_2) A_{\pi_{21}, \pi_{22}}(x_3; y_2, y_3) f(y_1, y_2, y_3) \right)^{1/p} \\ &= \left(\int_{\Delta_i} d^6 x \int_{\Delta_j} d^6 y A_{\pi_{21}, \pi_{22}}(x_3; x_2, y_3) f(x_1, x_2, y_3) \right)^{1/p} \\ &\leq \left(\int_{\Delta_i \times \Delta_j} d^6 x d^6 y |A_{\pi_{21}, \pi_{22}}(x_3; x_2, y_3)|^p \int f(x_1, x_2, y_3)^p \right)^{1/p} \\ &\leq \sup_{\substack{\lambda \in \Delta_i \\ y \in \Delta_j}} \left(\int d^6 x_3 |A_{\pi_{21}, \pi_{22}}(\lambda_3; x_2, y_3)|^p \int d^6 x_1 d^6 x_2 d^6 y_3 f(x_1, x_2, y_3)^p \right)^{1/p} \\ &= |A_{\pi_{21}, \pi_{22}}|_{p,1}^{(0)} \cdot \| f \|_{L_p} \end{aligned}$$

b) Let c, K_1 be positive constants such that

$$|A_{\Pi, \Pi'}|_{p,p}^{(c)} < K_1 \quad \text{for all partitions } \Pi, \Pi' \text{ of } \{1, \dots, n\}.$$

To obtain a convergent Neumann series

$$(1_n + \lambda A)^{-1} = 1_n + \sum_{m=1}^{\infty} (-\lambda A)^m \quad \text{it suffices to bound}$$

$$|(A^m)_{\Pi_0, \Pi_m}|_{p,p}^{(c/2)} \text{ by } K^m \text{ for some } K < \infty. \text{ By using}$$

$$d(i, k) \leq d(i, j) + d(j, k) \quad \text{we get}$$

$$\begin{aligned} |(A^m)_{\Pi_0, \Pi_m}|_{p,p}^{(c/2)} &\leq \sup_{j^0, j^m \in \mathbb{Z}^{2n}} \sum_{\Pi_0, \dots, \Pi_{m-1}} \sum_{\substack{j^1, \dots, j^{m-1} \\ j^e \in \mathbb{Z}^{2/\pi_e!}}} \prod_{e=1}^m \|P_{\Delta_{j^{e-1}}} A_{\pi_{e-1}} P_{\Delta_{j^e}}\|_{p,p} \\ &\leq \sup_{j^0, j^m} e^{c/2 d(j^0, j^m)} \sum_{\{\pi_e\}} \sum_{\{j^e\}} \prod_{e=1}^m K_1 e^{-cd(j^{e-1}, j^e)} \\ &\leq K_1^m \sum_{\{\pi_e\}} \sup_{j^0, j^m} \sum_{\{j^e\}} \prod_{e=1}^m e^{-c/2 d(j^{e-1}, j^e)} \\ &\leq K_1^m \sum_{\{\pi_e\}} \prod_{e=1}^m \left(\sup_{j^e \in \mathbb{Z}^{2/\pi_e!}} \sum_{j \in \mathbb{Z}^{2/\pi_e!}} e^{-c/2 d(i, j)} \right) \leq K^m. \end{aligned}$$

This completes the proof.

□

II THE N-PARTICLE IRREDUCIBLE KERNELS S^{N+1} AND THE BETHE-SALPETER KERNEL

We introduce the notation

$$S(\emptyset) = 1$$

$$S(X) = S(x_1, \dots, x_k)$$

$$S_{k,\ell}^1(x; y) = S(x; y) = S(X \cup Y) - S(X)S(Y) \quad ,$$

where $X = \{x_1, \dots, x_k\} \in \mathbb{R}^{2k}$, $Y = \{y_1, \dots, y_\ell\} \in \mathbb{R}^{2\ell}$.

The function $S_{k,\ell}^1(x; y)$ can be regarded as the kernel of an operator $S_{k,\ell}^1$ (see Propositions 4 b) and 8 b)). We now give a recursive definition of the n-particle irreducible kernel. A motivation of this definition will be given below.

Definition 6 :

$$S_{k,\ell}^{n+1} = S_{k,\ell}^n - S_{k,n}^n (S_{n,n}^n)^{-1} S_{n,\ell}^n \quad u = 1, 2, 3, \dots$$

An immediate consequence of this definition is

$$(II.1) \quad S_{k,\ell}^n = 0 \quad \text{if} \quad \min(k, \ell) < n$$

Remark : In perturbation theory $\equiv \bigcirc_n \equiv (X; Y) = S_{k,\ell}^n(X; Y)$ should be the collection of all graphs with at least n lines hitting each line separating the points of X from those of Y .

Definition 7 :

Let $\phi_n(f) = \int d^{2n} x f(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n)$ and $\mathcal{E}$ denote the completion of the span of $\{1, \phi_1(f_1), \phi_2(f_2), \dots : f_n \in S(\mathbb{R}^{2n})\}$ with respect to the scalar product $\langle A, B \rangle^0 = \langle AB \rangle$. Then with $P_0 A = A - \langle A \rangle$ we define P_n recursively as the orthogonal projection onto the subspace $\mathcal{E}_n$ of $\mathcal{E}$ spanned by the polynomials $(1 - \sum_{m < n} P_m) \phi_n(f)$.

Furthermore let the n -l-particle irreducible expectation $\langle A, B \rangle^n$ be defined by the linear extension of

$$\langle \phi_k(f), \phi_e(g) \rangle^n = \int d^{2k} x d^{2e} y f(x) S_{k,e}^n(x; y) g(y) .$$

Lemma 7 :

- a) $P_n P_m = \delta_{nm} P_m$
 b) For any $N \in \mathbb{N}$ there is a $\lambda(N) > 0$ such that for $n < N$,
 $0 \leq \lambda \leq \lambda(N)$ and $A, B \in \mathcal{E}$ the following is valid: $\langle A, B \rangle^n$
 $= \langle A (1 - \sum_{m < n} P_m) B \rangle$

In order to prove Lemma 7 we will first expand the kernels $S_{k,e}^n$ in terms of partially amputated Greens functions

$$S^T(X \cup Y) = S(x_1, \dots, x_k; y_1, \dots, y_e) .$$

Let the integral operators $\sigma_{k,e}^1$ be defined by their kernels

$$(II.4) \quad \sigma_{|X|,|Y|}^1(X;Y) = (|X|!)^{-1} \sum_{\substack{x_1 \cup \dots \cup x_k = X \\ y_1 \cup \dots \cup y_k = Y \\ x_i, y_i \neq \emptyset}} \frac{1}{k!} S^T(X_1 \cup Y_1) \dots S^T(X_k \cup Y_k),$$

and in analogy to Definition 6 let

$$\sigma_{k,e}^{n+1} = \sigma_{k,e}^n - \sigma_{k,n}^n (\sigma_{n,n}^n)^{-1} \sigma_{n,e}^n \quad \text{for } n=1,2,\dots.$$

Lemma 8 :

Let $N \in \mathbb{N}$ and $1 < p < \infty$ be given. Then for $n \leq N$ and λ, m_0^{-2} sufficiently small

$$a) \quad \sigma_{k,e}^n, (\sigma_{n,n}^n)^{-1} \in A_{p,p}^{*,*} \quad \forall k, e \in \mathbb{N}$$

$$b) \quad \sigma_{|X|,|Y|}^n(X;Y) = \sum_{\substack{X' \subset X \\ |X'| \geq n}} \sum_{\substack{Y' \subset Y \\ |Y'| \geq n}} S(X \setminus X') C_{|X'|} \sigma_{|X'|,|Y'|}^n(X';Y') S(Y \setminus Y')$$

Proof :

$$a) \quad \text{Let } E(X;Y) = \begin{cases} \delta(X-Y) & \text{if } X=\{x\}, Y=\{y\} \\ 0 & \text{otherwise} \end{cases}$$

Then by Proposition 4 b) $A(X;Y) = S^T(X \cup Y) - E(X;Y)$ defines an element $A \in A_{p,1}^{*,*}$ if λ is sufficiently small. By inserting

$S^T(X_i \cup Y_i) = E(X_i; Y_i) + A(X_i; Y_i)$ into the right hand side of (II.4) we obtain

$$(II.5) \quad G_{k,e}^1 = A_{k,e} + \sum_{m=1}^{\min(k,e)-1} 1_m \otimes A_{k-m,e-m} + \delta_{k,e} 1_e$$

where the $A_{i,j}$ are linear combinations of tensor products of elements $A \in A_{p,1}^{*,*}$. By Proposition 5 $A_{i,j} \in A_{p,1}^{*,*}$ and thus by Proposition 6 a) $\sigma_{k,e}^1 \in A_{p,p}^{*,*}$. Since $\sigma_{k,e}^1 = \delta_{k,e} 1_e + O(\lambda)$ it follows from Proposition 6 b) that for sufficiently small $\lambda > 0$ also

$$(\sigma_{1,1}^1)^{-1}, \sigma_{k,e}^2, \dots, \sigma_{k,e}^N, (\sigma_{n,n}^n)^{-1} \in A_{p,q}^{*,*}.$$

b) Let $S^T(X) = S(x_1; x_2; \dots; x_k)$. The definition of truncation

$$S(X \cup Y) = \sum_{\substack{w_1 \cup \dots \cup w_m = X \cup Y \\ w_i \neq \emptyset}} \frac{1}{m!} S^T(w_1) \dots S^T(w_m)$$

can be rewritten as

$$S(X \cup Y) = \sum_{\substack{x_1 \cup \dots \cup x_\lambda \cup z_1 \cup \dots \cup z_k \cup y_1 \cup \dots \cup y_\mu = X \cup Y \\ x_j, y_i, z_j \neq \emptyset, \lambda + k + \mu \neq 0 \\ x_j \cap y = \emptyset, y_j \cap x = \emptyset \\ z_j \cap y \neq \emptyset, z_j \cap x \neq \emptyset}} \frac{1}{\lambda! k! \mu!} S^T(x_1) \dots S^T(y_\mu).$$

If we subtract $S(X)S(Y)$ we get the same expression for $S(X;Y)$ but with the additional condition that $k \neq 0$.

By doing then the sum over X_j and Y_j one obtains

$$\begin{aligned} S(X;Y) &= \sum_{\substack{X' \subset X \\ X' \neq \emptyset}} \sum_{\substack{Y' \subset Y \\ Y' \neq \emptyset}} S(X \cdot X') S(Y \cdot Y') \sum_{\substack{Z_1 \cup \dots \cup Z_k = X' \cup Y' \\ Z_j \cap Y' \neq \emptyset, Z_j \cap X' \neq \emptyset}} \frac{1}{k!} S^T(Z_1) \dots S^T(Z_k) \\ &= \sum_{\substack{X' \subset X \\ X' \neq \emptyset}} \sum_{\substack{Y' \subset Y \\ Y' \neq \emptyset}} S(X \cdot X') S(Y \cdot Y') T_{|X'|, |Y'|}^1(X'; Y') \end{aligned}$$

where $T_{k,e}^1 = C_{k\sigma}^1$. So the assertion follows for $n = 1$. Let us now define

$$T_{k,e}^{n+1} = T_{k,e}^n - T_{k,n}^n (T_{n,n}^n)^{-1} T_{n,e}^n$$

and assume that for $m = n$ we have already shown

$$(II.6) \quad S_{|X'|, |Y'|}^m(X; Y) = \sum_{\substack{X' \subset X \\ |X'| \geq n}} \sum_{\substack{Y' \subset Y \\ |Y'| \geq n}} S(X \cdot X') S(Y \cdot Y') T_{|X'|, |Y'|}^m(X'; Y') .$$

Then for $|X| = n$ or $|Y| = n$ the first or the second sum reduces to a single term and $T_{n,n}^n = S_{n,n}^n$. Thus

$$\begin{aligned} S_{|X'|, |Y'|}^{n+1}(X; Y) &= [S_{|X'|, |Y'|}^n - S_{|X'|, n}^n (S_{n,n}^n)^{-1} S_{n, |Y'|}^n](X; Y) \\ &= \sum_{\substack{X' \subset X \\ |X'| \geq n}} \sum_{\substack{Y' \subset Y \\ |Y'| \geq n}} S(X \cdot X') S(Y \cdot Y') T_{|X'|, |Y'|}^{n+1}(X'; Y') . \end{aligned}$$

By definition $T_{k,\ell}^{n+1} = 0$ if $k = n$ or $\ell = n$ such that the sums reduce to $|x'|, |y'| \geq n+1$ and (II.6) follows for $m = n+1$.

Finally $T_{k,\ell}^n = C_{k,\ell} \sigma_{k,\ell}^n$ for $n = 1$ extends recursively to higher terms.

□

Proof of Lemma 7 :

a) follows immediately from the definition.

b) let the dense subspace $\mathcal{E}^0$ of $\mathcal{E}$ be defined as the linear hull of $\{1, \phi_\ell(f_\ell) : \ell \in \mathbb{N}, f \in L_2^0(\mathbb{R}^{2\ell})\}$ where $L_2^0(\mathbb{R}^{2\ell})$ denotes the subspace of $L_2(\mathbb{R}^{2\ell})$ of functions with compact support.

First we want to define linear operators $Q_0, Q_1, \dots, Q_N$ on $\mathcal{E}^0$, $Q_0 = P_0$, such that formally

$$(II.7) \quad \alpha_n \phi_\ell(f_\ell) = (1 - \sum_{m < n} P_m) \phi_n(S_{\eta_n}^{-1} S_{\eta_\ell} f_\ell) = Q_n \phi_n(S_{\eta_n}^{-1} S_{\eta_\ell} f_\ell).$$

For every partition $\Pi = \{\Pi^1, \dots, \Pi^{|\Pi|}\}$ of $\{1, \dots, n\}$ let

$f_{n,\ell,\Pi}$ be the Π -component (see Definition 3) of

$$f_{n,\ell} = \int d^{2\ell} y \sum_{\substack{y' \in \{y_1, \dots, y_\ell\} \\ |y'| \leq n}} (S_{\eta_n}^{-1})^{-1} S_{\eta_\ell}(\cdot, y') S(y, y') f_\ell(y).$$

By Lemma 8 $f_{n,\ell,\Pi} \in L_{\infty,2}(H_n^\Pi)$ with

$$f_{n,\ell,\Pi}(x_1, \dots, x_{|\Pi|}) = O\left(e^{-\text{const.} \sum_{i=1}^{|\Pi|} \text{dist}(x_i, \text{supp } f_\ell)}\right)$$

By using the notation $\phi_{n,\ell,\Pi}(x_1, \dots, x_{|\Pi|}) = \prod_{i=1}^{|\Pi|} \phi_{\eta_i}^{i_{\eta_i}}(x_i)$

we define

$$(II.8) \quad Q_n \phi_e(f_e) = (1 - \sum_{m < n} P_m) \sum_{\tau} \phi_{n,\tau}(f_{n,e,\tau}) .$$

Since by Lemma 1 $\langle \phi_{n,\tau_1}(\cdot) \phi_{n,\tau_2}(\cdot) \rangle \in L_{\infty,2}(\mathbb{R}^{2(l(\tau_1)+l(\tau_2))})$
it follows that for $f_e \in L_2^0(\mathbb{R}^{2e})$ $Q_n \phi_e(f_e) \in \mathcal{E}$, i.e.

$$\begin{aligned} & \| \phi_{n,\tau}(f_{n,e,\tau}) \| \\ &= \int d^{2l(\tau)} x d^{2l(\tau)} y f_{n,e,\tau}(x) \overline{\langle \phi_{n,\tau}(x) \phi_{n,\tau}(y) \rangle} f_{n,e,\tau}(y) < \infty . \end{aligned}$$

We may now assume that for $m = 1, 2, \dots, n$

$$\langle A, B \rangle^m = \langle A (1 - \sum_{j < m} P_j) B \rangle \quad \forall A, B \in \mathcal{E}$$

and that $Q_m - 1$ has the unique extension $\overline{Q_m - 1} = P_m - 1$.

Then using Lemma 8 we obtain

$$\begin{aligned} (II.9) \quad & \langle \phi_k(f_k), Q_n \phi_e(f_e) \rangle^0 \\ &= \sum_{\tau} \langle \phi_k(f_k) (1 - \sum_{m < n} P_m) \phi_{n,\tau}(f_{n,e,\tau}) \rangle^0 \\ &= \sum_{\tau} \langle \phi_k(f_k), \phi_{n,\tau}(f_{n,e,\tau}) \rangle^n \\ &= \sum_{\tau} \int d^{2l(\tau)} z (S_{n,k}^n f_k)_{\tau}(z) ((S_{n,n}^n)^{-1} S_{n,e}^n f_e)_{\tau}(z) \\ &= \int d^{2k} x d^{2e} y f_k(x) [S_{k,n}^n (S_{n,n}^n)^{-1} S_{n,e}^n](x; y) f_e(y) \end{aligned}$$

and it follows that Q_n is symmetric.

Furthermore, when restricted to $\bigoplus_{j \leq n} \mathcal{E}_j$, Q_n has a unique extension which coincides with $P_n \upharpoonright_{\bigoplus_{j \leq n} \mathcal{E}_j}$. This follows since $Q_n \phi_j(f_j) = 0$ if $j < n$. These properties together with the fact that the range of Q_n is contained in $\mathcal{E}_n$ ensure that $\overline{Q}_n = P_n$.

Finally (II.3) is obtained from (II.9) and from Definition 6.

Definition 8 :

For given $1 < p < q < \infty$ we define

$k : L_q^1 \rightarrow L_p^{1*}$ and $K : L_q^2 \rightarrow L_p^{2*}$ to be the solutions of

$$(II.10) \quad S_{1,1}^1 = C_1 - C_1 k S_{1,1}^1$$

and of the Bethe-Salpeter equation

$$(II.11) \quad S_{2,2}^2 = 2(S_{1,1}^1 \otimes S_{1,1}^1) - 2(S_{1,1}^1 \otimes S_{1,1}^1) K S_{2,2}^2.$$

These equations make sense since

$C_1, S_{1,1,1}^1, S_{1,1,1}^1 \otimes S_{1,1,1}^1, S_{2,2}^2 \in A_{q,p}^{1,*}$ by Propositions 4 a) and 5 and Lemma 8.

Proposition 9 : For sufficiently small $\lambda > 0$ (II.10) and (II.11) have unique

solutions $k, K \in A_{p,q}^{*}$ and

$$a) \quad k = (S_{1,1}^1)^{-1} - C^{-1}$$

$$b) \quad K = (S_{2,2}^2)^{-1} - \frac{1}{2}(S_{1,1}^1 \otimes S_{1,1}^1)^{-1}$$

on dense subsets of L_p^1, L_p^2 respectively.

Proof :

a) Let $A_{k,e}^1 \in A_{p,q}^{*}$ be defined by the kernel

$$A_{k,e}^1(x_1, \dots, x_k; y_1, \dots, y_e) = S(x_1, \dots, x_k; y_1, \dots, y_e) .$$

Then $\sigma_{1,1}^1 = 1_1 + A_{1,1}^1 C_1$. Furthermore $\sigma_{1,1}^1 = 1_1 - k C_1 \sigma_{1,1}^1$

by using (II.10) and the fact that $\mathcal{D}_1 = C_1 L_p^{1*}$ is dense in

L_q^1 . We may multiply by $(\sigma_{1,1}^1)^{-1} \in A_{p,p}^{*,*}$ from the right and

obtain

$$(II.12) \quad k C_1 = (S_{1,1}^1)^{-1} - 1_1 = -(S_{1,1}^1)^{-1} A_{1,1}^1 C_1$$

and thus

$$k = (S_{1,1}^1)^{-1} - C_1^{-1} \text{ on } \mathcal{D}_1 \text{ with}$$

$$k = -(S_{1,1}^1)^{-1} A_{1,1}^1 \in A_{p,q}^{*} .$$

b) Using that $\mathcal{D}_2 = C_2 L_q^{2*}$ is dense in L_q^2 is follows from

(II.11) that

$$S_{2,2}^2 = S_{1,1}^1 \otimes S_{1,1}^1 - (S_{1,1}^1 \otimes S_{1,1}^1) K C_2 S_{2,2}^2 .$$

Thus by defining

$$\begin{aligned} A_{k,e}^2 &= A_{k,e}^1 - A_{k,1}^1 C_1 (\bar{G}_{1,1}^1)^{-1} A_{1,e}^1 \\ \alpha &= (\bar{G}_{1,1}^1 \otimes \bar{G}_{1,1}^1)^{-1} A_{2,2}^2 C_2 \\ &= ((1_1 + k C_1) \otimes (1_1 + k C_1)) A_{2,2}^2 C_2 \quad \text{by (I.12)} \\ \beta &= ((1_1 + k C_1) \otimes (1_1 + k C_1)) A_{2,2}^2 ((1_1 + C_1 k) \otimes (1_1 + C_1 k)), \end{aligned}$$

KC_2 can be written as

$$\begin{aligned} \text{(II.13)} \quad KC_2 &= (\bar{G}_{2,2}^2)^{-1} - (\bar{G}_{1,1}^1 \otimes \bar{G}_{1,1}^1)^{-1} \\ &= (\bar{G}_{1,1}^1 \otimes \bar{G}_{1,1}^1 + A_{2,2}^2 C_2)^{-1} - (\bar{G}_{1,1}^1 \otimes \bar{G}_{1,1}^1)^{-1} \\ &= ((1_2 + \alpha)^{-1} - 1_2) (\bar{G}_{1,1}^1 \otimes \bar{G}_{1,1}^1)^{-1} \\ &= -(1_2 + \alpha)^{-1} \alpha (\bar{G}_{1,1}^1 \otimes \bar{G}_{1,1}^1)^{-1} = -(1_2 + \alpha)^{-1} \beta C_2 \end{aligned}$$

where we have used that

$$\bar{G}_{2,2}^2 = \bar{G}_{1,1}^1 \otimes \bar{G}_{1,1}^1 + A_{2,2}^2 C_2. \quad \text{Notice that } A_{2,2}^2 \in A_{p',q}^*,$$

since $C_1 (\bar{G}_{1,1}^1)^{-1} \in A_{q,p}^*$ and that $C_2 \in A_{q,p}^*$.

$$\text{Thus } \alpha \in A_{pp}^{*,*} \text{ and } \beta \in A_{p,q}^{*,*} \text{ if (II.14) and (II.15)}$$

below are valid. The steps which lead to (II.14) are

$$C_1 \in A_{q,1}^{*,*} \quad \text{by Proposition 4 a)}$$

$$kC_1 \in A_{p,1}^{*,*} \quad \text{by Proposition 9 a)}$$

p can be replaced by $2p$ for small λ . So

$$\text{(II.14)} \quad (1_1 + k C_1) \otimes (1_1 + k C_1) \in A_{p,p}^{*,*}$$

by the Propositions 5 and 6.

It is easy to see that in an analogous way one can prove

$$C_1 \in A_{\infty, p}^{'*}$$

$$C_1 k \in A_{\infty, q}^{'}$$

$$(I.15) \quad (1_1 + C_1 k) \otimes (1_1 + C_1 k) \in A_{q, q}^{'}$$

Now from (II.13) and $\bar{D}_2 = I_P^2$ we can deduce

$$K = (S_{2,2}^2)^{-1} - \frac{1}{2} (S_{1,1}^1 \otimes S_{1,1}^1)^{-1} \text{ on } D_2 \text{ with}$$

$$K = -(1_2 + \alpha)^{-1} \cdot \beta \in A_{pq}^{'*} \text{ as unique extension .}$$

□

Definition 9 :

Let

$$e_{\lambda, k, p}(x) = \chi_{\lambda}(x) \exp[-i(\frac{k}{2} + p)x] .$$

Then the one particle irreducible two point function $k(\lambda; v)$

and the Bethe-Salpeter kernel $K(\lambda; k, p, q)$ are defined by

$$k(\lambda; v) = \lim_{\lambda \rightarrow R^2} |\lambda|^{-1} \langle e_{\lambda, 0, v}, k e_{\lambda, 0, v} \rangle_1$$

$$K(\lambda; k, p, q) = (2\pi)^{-1} \lim_{\lambda \rightarrow R^2} |\lambda|^{-1} \langle e_{\lambda, k, -p} \otimes e_{\lambda, k, p}, k e_{\lambda, k, -q} \otimes e_{\lambda, k, q} \rangle_2$$

Finally in analogy to $\sigma_{k, e}^n$ let us define

$$\bar{\sigma}_{k, \lambda p}^1(x_1, \dots, x_k; \gamma) = -S(x_1; \dots; x_k; \gamma)$$

$$\bar{\sigma}_{k, \lambda p}^{n+1} = \bar{\sigma}_{k, \lambda p}^n - \bar{\sigma}_{k, n}^n (\bar{\sigma}_{n, n}^n)^{-1} \bar{\sigma}_{n, \lambda p}^n \quad \text{for } n=1, 2, \dots$$

Proposition 10 : Let $L = (\sigma_{2,2}^2)^{-1} \sigma_{2,\lambda}^2$. Then for $1 < p < q < \infty$
and sufficiently small λ $L \in A_{p,q}^{*,1}$

Proof :

This follows from Proposition 4 b) and Lemma 8 a)

□

III DERIVATIVES WITH RESPECT TO t

Our goal is to show that for certain objects A , namely the kernels defined in Definition 6, (II.4), Definition 8 and Proposition 10 one has $d_t^\alpha A|_t = 0$ for $\alpha \leq \alpha(A)$.

In this section we only consider the dependence on one of the parameters $t_i \in t$. $t \setminus \{t_i\}$ is set fixed and we shall write t instead of t_i . The starting point is the formula for t -derivatives [S]

$$(III.1) \quad \partial_t S(x_1, \dots, x_n) = -\frac{1}{2} \int d^4 z (\partial_t C^{-1})(t, z^1, z^2) S(x_1, \dots, x_n; z^1, z^2)$$

which becomes plausible if one writes formally

$$(III.2) \quad S(x_1, \dots, x_n) = \frac{\int \phi(x_1) \dots \phi(x_n) e^{-\frac{1}{2} \int d^4 z \phi(z^1) C^{-1}(t, z^1, z^2) \phi(z^2) - \lambda \int d^4 x : \phi(x) : (x)} d\phi}{\int e^{-\frac{1}{2} \int d^4 z \phi(z^1) C^{-1}(t, z^1, z^2) \phi(z^2) - \lambda \int d^4 x : \phi(x) : (x)} d\phi}$$

Proposition 11 :

$$\partial_t S(\dot{x}_1^1, \dots, \dot{x}_{m_1}^1, \dot{y}_1^1, \dots, \dot{y}_{n_1}^1; \dots; \dot{x}_1^k, \dots, \dot{x}_{m_k}^k, \dot{y}_1^k, \dots, \dot{y}_{n_k}^k) \\ = \frac{1}{2} \int d^4 z \dot{C}(\dot{z}^1, \dot{z}^2) S(\dot{x}_1^1, \dots, \dot{x}_{m_1}^1, \dot{y}_1^1, \dots, \dot{y}_{n_1}^1; \dots; \dot{x}_1^k, \dots, \dot{x}_{m_k}^k, \dot{y}_1^k, \dots, \dot{y}_{n_k}^k; \dot{z}^1, \dot{z}^2)$$

Proof :

We consider first the case without truncation ($k = 1$). Let $X = \{\dot{x}^1, \dots, \dot{x}^m, \dot{x}^{m+1}, \dots, \dot{x}^{m+n}\}$. Then from (III.1) it follows easily that for $m = 0$

$$(III.3) \quad \partial_t S(X) = \frac{1}{2} \int d^4 z \dot{C}(\dot{z}^1, \dot{z}^2) S(X; \{\dot{z}^1, \dot{z}^2\})$$

Then by using the notation $X^+ = \{\dot{x}^1, \dots, \dot{x}^{m+1}, \dot{x}^{m+2}, \dots, \dot{x}^{m+n}\}$ and

$$(III.4) \quad S(X^+) = \int d^2 y \dot{C}^{-1}(\dot{x}^{m+1}, y) [S(\{y\} \cup X \setminus \{\dot{x}^{m+1}\}) \\ - \sum_{j=1}^m S(X \setminus \{\dot{x}^j, \dot{x}^{m+1}\}) \delta(\dot{x}^j - y)]$$

we obtain

$$\partial_t S(X^+) = \int d^2 y \dot{C}^{-1}(\dot{x}^{m+1}, y) \int d^4 z \frac{1}{2} \dot{C}(\dot{z}^1, \dot{z}^2) \\ \times [S(\{y\} \cup X \setminus \{\dot{x}^{m+1}\}; \{\dot{z}^1, \dot{z}^2\}) - \sum_{j=1}^m S(X \setminus \{\dot{x}^j, \dot{x}^{m+1}\}; \{\dot{z}^1, \dot{z}^2\}) \delta(\dot{x}^j - y)] \\ - \int d^2 y \dot{C}^{-1}(\dot{x}^{m+1}, y) \int d^4 z \dot{C}(\dot{z}^1, \dot{z}^2) [S(\{\dot{z}^1\} \cup X \setminus \{\dot{x}^{m+1}\}) \delta(\dot{z}^2 - y)]$$

$$\begin{aligned}
 &= \frac{1}{2} \int d^4 z \dot{C}(z^1, z^2) \int d^2 y \dot{C}^{-1}(x^{m+1}, y) [S(\{y\} \cup X \setminus \{x^{m+1}\}) \delta(z^1, z^2) \\
 &\quad - \sum_{j=1}^m S(X \setminus \{x^j, x^{m+1}\}) \delta(z^1, z^2) \delta(x^j - y) \\
 &\quad - S(\{z^1\} \cup X \setminus \{x^{m+1}\}) \delta(z^2 - y) - S(\{z^2\} \cup X \setminus \{x^{m+1}\}) \delta(z^1 - y)] \\
 &= \frac{1}{2} \int d^4 z \dot{C}(z^1, z^2) S(X^+; \{z^1, z^2\}) .
 \end{aligned}$$

This proves (III.3) for $m \leq n$.

For subsets Π_j of a partition $\Pi = \{\Pi^1, \dots, \Pi^{|\Pi|}\}$ of a set $X = \{x^1, \dots, x^m, x^{m+1}, \dots, x^{m+n}\}$ let $X_{\Pi_j} = \{x : x \in \alpha \text{ for some } \alpha \in \Pi_j\}$. Then partially truncated functions S^Π can be defined as follows

$$(\text{III.5}) \quad S^\Pi(X) = - \sum_{\substack{\Pi_1 \cup \dots \cup \Pi_k = \Pi \\ \Pi_j \neq \emptyset}} \frac{(-1)^k}{k} S(X_{\Pi_1}) \dots S(X_{\Pi_k}) .$$

By using (III.2) we obtain

$$\begin{aligned}
 \partial_t S^\Pi(X) &= \frac{1}{2} \int d^4 z \dot{C}(z^1, z^2) \\
 &\times \sum_{\substack{\Pi_1 \cup \dots \cup \Pi_k = \Pi \\ \text{ordered}}} (-1)^{k-1} (k-1)! \sum_{e=1}^k [S(X_{\Pi_e} \cup \{z^1, z^2\}) - S(X_{\Pi_e}) S(z^1, z^2)] \frac{k}{j \neq e} S(X_{\Pi_j}) \\
 &= \frac{1}{2} \int d^4 z \dot{C}(z^1, z^2) \sum_{\substack{\Pi_1 \cup \dots \cup \Pi_k = \Pi \cup \{z^1, z^2\} \\ \text{ordered}}} (-1)^{k-1} (k-1)! S(X \cup \{z^1, z^2\})_{\Pi_1} \dots S(X \cup \{z^1, z^2\})_{\Pi_k} \\
 &= \frac{1}{2} \int d^4 z \dot{C}(z^1, z^2) S^{\Pi \cup \{z^1, z^2\}}(X \cup \{z^1, z^2\}) .
 \end{aligned}$$

□

Before applying Proposition 11 to compute derivatives of the kernels defined in the previous section we need

Proposition 12 : Let K_t denote one of the operators
 $C_n, S, \sigma_{k,\ell}^n, k, K, L$ and $A_{pq}^{\sigma\tau}$ the class to which K_t belongs
according to Lemma 4, Propositions 8, 9 and 10. Then
 $d_t^\alpha K_t \in A_{p,q}^{\sigma\tau}$ for $t \in [0,1]$.

Proof :

First notice that the previous estimates for $K_{t,h}$ are uniform in $t \in [0,1]$ and $h \in \mathbb{R}$ where $K_{t,h}$ is the operator constructed from the expectation $\langle \dots \rangle_{t,h}$. By Proposition 1 these expectations are analytic in h for $h \in \mathbb{R}$ such that $K_{t,h}$ as absolutely convergent Neumann series is also analytic by Fubini's theorem. Using the definition of $K_{t,h}$ we have

$$\partial_t^\alpha K_t = \partial_h^\alpha K_{t,h} \Big|_{h=0} ,$$

that is,

$$\partial_t^\alpha K_t = \frac{\alpha!}{2\pi i} \oint_{|h|=1} \frac{dh}{h^{\alpha+1}} K_{t,h} .$$

The assertion follows now easily. □

For the following calculations we make some notational simplifications :

- a) If $\chi_e(\chi_r)$ denotes the characteristic function on $\mathbb{R}^2$ of the half plane on the left (right) side of the t-line then

$$A(\dots, \overset{e}{x}, \dots) = A(\dots, x, \dots) \chi_e(x) \Big|_{t=0}$$

$$A_e(x_1, \dots, x_n) = A(\overset{e}{x}_1, \dots, \overset{e}{x}_n)$$

and analogously with r .

- b) Distinguishing between variables z and others we write

$$A_1(x_1, \dots, x_k, y_1, \dots, y_n) A_2(y_1, \dots, y_n, y_1, \dots, y_e)$$

$$\int d^{2n} y A_1(x_1, \dots, x_k, y_1, \dots, y_n) A_2(y_1, \dots, y_n, y_1, \dots, y_e) ,$$

$$A(x_1, \dots, x_k, z^1, z^2)$$

$$- \frac{1}{2} \int d^4 z \dot{C}(z^1, z^2) A(x_1, \dots, x_k, z^1, z^2) .$$

- c) Let $\sigma^n(X; Y) = \sigma^n_{|X|, |Y|}(X; Y)$, $\sigma(X; Y) = \sigma^1(X; Y)$ and $\underline{x} = x_1; x_2; \dots; x_k$ if $X = \{x_1, \dots, x_k\}$.

- d) $\dot{f} = d_t f$, $\ddot{f} = d_t^2 f, \dots$

The reason for which the parameter t was introduced is that the measure $d_{m_0^2}(t=0)$ decouples along the line l , i.e.

$C(t = 0, x, y) = C(0, \overset{e}{x}, \overset{e}{y}) + C(0, \overset{r}{x}, \overset{r}{y})$, such that

$$(III.6) \quad S(\overset{e}{X}, \overset{r}{Y}) = S(\overset{e}{X}) S(\overset{r}{Y})$$

(see (III.2)). This factorization property will now be applied to kernels of increasing complexity.

Some immediate consequences are listed in

Proposition 13 : a) $S(\overset{e}{X}, \overset{r}{Y}) = 0$

$$b) \quad S(\overset{e}{X}, \overset{r}{Y}) = S_e(\overset{e}{X}) S_r(\overset{r}{Y})$$

$$c) \quad S(\overset{e}{X}; \overset{e}{z}_1, \overset{e}{z}_2; \overset{r}{z}_1, \overset{r}{z}_2; \overset{r}{Y})$$

$$= 8 \sum_{x_1 \cup x_2 = X} S_e(x_1; \overset{e}{z}_1) S_e(x_2; \overset{e}{z}_2) S_r(\overset{r}{z}_1; \overset{r}{z}_2; \overset{r}{Y})$$

$$+ 8 \sum_{y_1 \cup y_2 = Y} S_e(\overset{e}{X}; \overset{e}{z}_1, \overset{e}{z}_2) S_r(\overset{r}{z}_1; y_1) S_r(\overset{r}{z}_2; y_2)$$

$$+ 4 S_e(\overset{e}{X}; \overset{e}{z}_1, \overset{e}{z}_2) S_r(\overset{r}{z}_1; \overset{r}{z}_2; \overset{r}{Y})$$

$$+ 4 S_e(\overset{e}{X}; \overset{e}{z}_1) S(\overset{e}{z}_2; \overset{e}{z}_1, \overset{e}{z}_2; \overset{r}{Y})$$

$$+ 4 S(\overset{e}{X}; \overset{e}{z}_1, \overset{e}{z}_2; \overset{e}{z}_1) S_r(\overset{r}{z}_2; \overset{r}{Y})$$

Analogous formulas are valid for partially amputated functions.

Proposition 14 : a) $\bar{S}(\{\overset{e}{x}'\} \cup \overset{r}{X}; \overset{r}{Y}) = 0$

$$b) \quad \bar{S}(\overset{e}{X}_1 \cup \overset{r}{X}_2; \overset{r}{Y}) = \frac{2^{(1+|X_2|)!}}{(|X_1|+|X_2|)!} S_e(\overset{e}{X}_1; \overset{e}{z}) \bar{S}_r(\overset{r}{z}_1, \overset{r}{z}_2; \overset{r}{Y})$$

$$\begin{aligned}
 c) \quad \ddot{G}(\overset{e}{X}; \overset{r}{Y}) &= 8 \, G_e(X; \{z_1^1, z_2^1\}) \, G_r(\{z_1^2, z_2^2\}; Y) \\
 &+ \frac{4}{|X|!} \left[S_e(\overset{e}{X}; \overset{1}{z}_1^1) S(\overset{1}{z}_1^2; \overset{1}{z}_2^1, \overset{1}{z}_2^2; \overset{r}{Y}) \right. \\
 &\quad \left. + S(\overset{e}{X}; \overset{1}{z}_1^1, \overset{1}{z}_1^2; \overset{1}{z}_2^1) S_r(\overset{1}{z}_2^2; Y) \right]
 \end{aligned}$$

This follows easily from (II.4) and Proposition 13.

Proposition 15 : a) $\dot{G}^{-1}(\overset{e}{X}; \overset{r}{Y}) = 0$

$$b) \quad \dot{G}^2(\overset{e}{X}; Y \cup \{Y'\}) = (\dot{G}^2)^{-1}(\overset{e}{x}_1, \overset{e}{x}_2; \overset{r}{y}_1, \overset{r}{y}_2) = 0$$

$$c) \quad \partial_t \dot{G}^2(\overset{e}{X}; \overset{r}{Y}) = \partial_t (\dot{G}^2)^{-1}(\overset{e}{x}_1, \overset{e}{x}_2; \overset{r}{y}_1, \overset{r}{y}_2) = 0$$

Proof :

$$b) \quad \dot{G}^2(\overset{e}{X}; Y \cup \{Y'\}) = \dot{G}(\overset{e}{X}; \overset{r}{Y} \cup \{Y'\}) - \dot{G}(\overset{e}{X}; \overset{r}{Y}) \dot{G}^{-1}(\overset{r}{Y}; \overset{r}{Y}') \dot{G}(\{Y'\}; \overset{r}{Y} \cup \{Y'\})$$

If p_e^n denotes the projection on L_p^{n*} defined by

$p^n f = f_e$, then p_e^2 commutes with $\sigma_{2,2}^2$. Hence it

commutes with $(\sigma_{2,2}^2)^{-1}$ and thus

$$p_e^2 (\sigma_{2,2}^2)^{-1} (1 - p_e^2) = (\sigma_{2,2}^2)^{-1} p_e^2 (1 - p_e^2) = 0.$$

a) By Proposition 14 a) $\sigma_{1,1}^1$ commutes with p_ϵ^1 . Thus
 $p_\epsilon^1 (\sigma_{1,1}^1)^{-1} (1 - p_\epsilon^1) = (\sigma_{1,1}^1)^{-1} p_\epsilon^1 (1 - p_\epsilon^1) = 0$.

c) $d_t(\sigma_{2,2}^2)^{-1} (x_1^\epsilon, x_2^\epsilon; y_1^r, y_2^r) = 0$ follows from Proposition 15 b) together with

$$|X|! \bar{G}^2(X; \overset{\epsilon}{Y})$$

$$= |X|! \partial_t [\bar{G}(\cdot; \cdot) - \bar{G}(\cdot; \gamma) \bar{G}^{-1}(\gamma; \gamma') \bar{G}(\gamma'; \cdot)] (X; \overset{\epsilon}{Y})$$

$$= S_e(\underline{x}; \underline{z}^{\epsilon'}) S_r(\underline{z}^2; \gamma)$$

$$= S_e(\underline{x}; \underline{z}^{\epsilon'}) S_r(\gamma; \underline{z}^2) \bar{G}_r^{-1}(\gamma; \gamma') S_r(\gamma'; \gamma)$$

$$+ S_e(\underline{x}; \gamma) \bar{G}_e^{-1}(\gamma; \gamma_1) S(\gamma_1^{\epsilon}; \gamma_2^{\epsilon}; \underline{z}^{\epsilon'}; \underline{z}^2) \bar{G}_r^{-1}(\gamma_2; \gamma') S_r(\gamma'; \gamma)$$

$$= S_e(\underline{x}; \gamma) \bar{G}_e^{-1}(\gamma; \gamma') S(\gamma_1^{\epsilon}; \underline{z}^{\epsilon'}; \underline{z}^2; \gamma)$$

$$= (1-1+1-1) \cdot 2 S_e(\underline{x}; \underline{z}^{\epsilon'}) S_r(\underline{z}^2; \gamma)$$

$$= 0$$

Proposition 16 : a) $\bar{G}^2(X \cup \{x'\}; \overset{\epsilon}{Y}) = \frac{4}{|X|+1} \bar{G}_e(X; \{\underline{z}^{\epsilon'}\}) \bar{G}_r^2(\{x', x'\}; \gamma)$

$$b) \bar{G}^2(X; \overset{\epsilon}{Y}) = 8 \bar{G}_e^2(X; \{\underline{z}^{\epsilon'}, \underline{z}^{\epsilon'}\}) \bar{G}_r^2(\{\underline{z}^2, \underline{z}^2\}; \gamma)$$

Proof :

a) Using Proposition 14 b)

$$\begin{aligned}
 & (|X|+1)! \, \bar{G}^2(X \cup \{x'\}; \bar{Y}) \\
 &= 4 S_e(\underline{x}; \bar{z}^1) \, \bar{G}_r(\{z^2, x'\}; Y) \\
 &- 2 S_e(\underline{x}; \bar{z}^1) S_r(\bar{z}^2; x'; \eta) \, \bar{G}_r^{-1}(\eta; \eta') S_r(\eta'; Y) \\
 &+ 2 S(\underline{x}; x'; \eta) \, [\dots] \\
 &= 4 |X|! \, \bar{G}_e(X; \{z^1\}) \, \bar{G}_r^{-2}(\{z^2, x'\}; Y) .
 \end{aligned}$$

b) Two terms which cancel each other are indicated by the same $\textcircled{n}$.

$$\begin{aligned}
 & |X|! \, \bar{G}^2(X; \bar{Y}) = 8 |X|! \, \bar{G}_e(X; \{\bar{z}_1^1, \bar{z}_2^1\}) \, \bar{G}_r(\{z_1^2, z_2^2\}; Y) \\
 &\textcircled{4} + 4 S_e(\underline{x}; \bar{z}_1^1) S(\bar{z}_1^2; \bar{z}_2^1, z_2^2; Y) \\
 &\textcircled{1} + 4 S(\underline{x}; \bar{z}_1^1, \bar{z}_2^1; \bar{z}_2^1) S_r(\bar{z}_2^2; Y) \\
 &- 8 |X|! \, \bar{G}_e(X; \{\bar{z}_1^1, \bar{z}_2^1\}) \, \bar{G}_r(\{z_1^2, z_2^2\}; \eta) \, \bar{G}_r^{-1}(\eta; \eta') S_r(\eta'; Y) \\
 &\textcircled{2} - 4 S_e(\underline{x}; \bar{z}_1^1) S(\bar{z}_1^2; z_2^1, z_2^2; \eta) \, \bar{G}_r^{-1}(\eta; \eta') S_r(\eta'; Y) \\
 &\textcircled{1} - 4 S(\underline{x}; \bar{z}_1^1, \bar{z}_2^1; \bar{z}_2^1) S_r(\bar{z}_2^2; \eta) \, \bar{G}_r^{-1}(\eta; \eta') S_r(\eta'; Y) \\
 &\textcircled{3} + 4 S(\underline{x}; \bar{z}_1^1, \bar{z}_2^1; \eta) \, \bar{G}_e^{-1}(\eta; \eta_1) S_e(\eta_1; \bar{z}_2^1) S_r(\bar{z}_2^2; \eta_2) \, \bar{G}_r^{-1}(\eta_2; \eta') \bar{G}_r(\eta'; Y) \\
 &\textcircled{2} + 4 S_e(\underline{x}; \bar{z}_1^1) S_r(\bar{z}_1^2; \eta) \, \bar{G}_r^{-1}(\eta; \eta_1) S(\eta_1; \bar{z}_2^1, \bar{z}_2^2; \eta_2) \, \bar{G}_r^{-1}(\eta_2; \eta') \bar{G}_r(\eta'; Y) \\
 &\textcircled{3} - 4 S(\underline{x}; \bar{z}_1^1, \bar{z}_2^1; \eta) \, \bar{G}_e^{-1}(\eta; \eta') S_e(\eta'; \bar{z}_2^1) S_r(\bar{z}_2^2; Y) \\
 &\textcircled{4} - 4 S_e(\underline{x}; \bar{z}_1^1) S_r(\bar{z}_1^2; \eta) \, \bar{G}_r^{-1}(\eta; \eta') S(\eta'; \bar{z}_2^1, \bar{z}_2^2; Y) \\
 &+ 8 |X|! \, S_e(\underline{x}; \eta) \, \bar{G}_e^{-1}(\eta; \eta_1) \bar{G}_e(\eta_1; \{\bar{z}_1^1, \bar{z}_2^1\}) \bar{G}_r(\{z_1^2, z_2^2\}; \eta_2) \, \bar{G}_r^{-1}(\eta_2; \eta') S_r(\eta'; Y)
 \end{aligned}$$

$$\begin{aligned}
 (5) & + 4 S_e(\underline{x}; \gamma) \bar{G}_e^{-1}(\gamma; \gamma_1) S_e(\gamma_1; \underline{z}_1^1) S(\underline{z}_1^2; \underline{z}_2^1, \underline{z}_2^2; \gamma_2) \bar{G}_r^{-1}(\gamma_2; \gamma) S_r(\gamma'; \gamma) \\
 (8) & + 4 S_e(\underline{x}; \gamma) \bar{G}_e^{-1}(\gamma; \gamma_2) S(\gamma_2; \underline{z}_1^1, \underline{z}_2^1; \underline{z}_2^2) S_r(\underline{z}_2^2; \gamma_2) \bar{G}_r^{-1}(\gamma_2; \gamma') S_r(\gamma'; \gamma) \\
 (6) & - 4 S_e(\underline{x}; \gamma) \bar{G}_e^{-1}(\gamma; \gamma_{11}) S(\gamma_{11}; \underline{z}_1^1, \underline{z}_2^1; \gamma_{12}) \bar{G}_e^{-1}(\gamma_{12}; \gamma_1) S_e(\gamma_1; \underline{z}_2^1) \\
 & \quad * S_r(\underline{z}_2^2; \gamma_2) \bar{G}_r^{-1}(\gamma_2; \gamma') S_r(\gamma'; \gamma) \\
 (5) & - 4 S_e(\underline{x}; \gamma) \bar{G}_e^{-1}(\gamma_1; \gamma_{11}) S_e(\gamma_{11}; \underline{z}_1^1) S_r(\underline{z}_2^2; \gamma_{12}) \bar{G}_r^{-1}(\gamma_{12}; \gamma_1) \\
 & \quad * S(\gamma_1; \underline{z}_2^1, \underline{z}_2^2; \gamma_2) \bar{G}_r^{-1}(\gamma_2; \gamma') S_r(\gamma'; \gamma) \\
 (6) & + 4 S_e(\underline{x}; \gamma) \bar{G}_e^{-1}(\gamma; \gamma_1) S(\gamma_1; \underline{z}_1^1, \underline{z}_2^1; \gamma_2) \bar{G}_e^{-1}(\gamma_2; \gamma') S_e(\gamma'; \underline{z}_2^1) S_r(\underline{z}_2^2; \gamma) \\
 (7) & + 4 S_e(\underline{x}; \gamma) \bar{G}_e^{-1}(\gamma; \gamma_1) S_e(\gamma_1; \underline{z}_1^1) S_r(\underline{z}_2^2; \gamma_2) \bar{G}_r^{-1}(\gamma_2; \gamma') S(\gamma'; \underline{z}_2^1, \underline{z}_2^2; \gamma) \\
 & - 8 |X|! S_e(\underline{x}; \gamma) \bar{G}_e^{-1}(\gamma; \gamma') \bar{G}_e^{-1}(\{\gamma'\}; \{\underline{z}_1^1, \underline{z}_2^1\}) \bar{G}_r^{-1}(\{\underline{z}_1^2, \underline{z}_2^2\}; \gamma) \\
 (7) & - 4 S_e(\underline{x}; \gamma) \bar{G}_e^{-1}(\gamma; \gamma') S_e(\gamma'; \underline{z}_1^1) S(\underline{z}_1^2; \underline{z}_2^1, \underline{z}_2^2; \gamma) \\
 (8) & - 4 S_e(\underline{x}; \gamma) \bar{G}_e^{-1}(\gamma; \gamma') S(\gamma'; \underline{z}_1^1, \underline{z}_2^1; \underline{z}_2^2) S_r(\underline{z}_2^2; \gamma) \\
 & = 8 |X|! \bar{G}_e^{-2}(X; \{\underline{z}_1^1, \underline{z}_2^1\}) \bar{G}_r^{-2}(\{\underline{z}_1^2, \underline{z}_2^2\}; \gamma)
 \end{aligned}$$

□

Proposition 17 : a) $\partial_e(\bar{G}_{\gamma_2}^{-1}) \bar{G}_{\gamma_2}^{-1}(x_1, x_2; \gamma) = \bar{L}(x_1, x_2; \gamma) = L(x_1, x_2; \gamma) = 0$

$$b) \partial_e^2(\bar{G}_{\gamma_2}^{-1}) \bar{G}_{\gamma_2}^{-1}(x_1, x_2; \gamma) = \bar{L}^{\cdot\cdot}(x_1, x_2; \gamma) = 0$$

Proof :

a) By Proposition 15 b), c) and Proposition 16 a)

$$\begin{aligned}
 & \partial_t (\bar{\sigma}_{2,2}^{-1}) \bar{\sigma}_{2,e}^{\epsilon, \epsilon, \epsilon} (x_1, x_2; \gamma) \\
 &= -2 (\bar{\sigma}_{2,2}^{-1}) (x_1, x_2; \gamma_1, \gamma_1') 2 S_e(\gamma_2; \bar{z}^1) \\
 &\quad \times (\bar{\sigma}_{2,2})_r(\bar{z}^1, \gamma_1'; \gamma_2, \gamma_2') (\bar{\sigma}_{2,2}^{-1}) (\gamma_2, \gamma_2'; \gamma_3, \gamma_3') (\bar{\sigma}_{2,e})_r(\gamma_3, \gamma_3'; \gamma) \\
 &\quad + 2 (\bar{\sigma}_{2,2}^{-1})_e(x_1, x_2; \gamma_1, \gamma_1') 2 S_e(\gamma_2; \bar{z}^1) (\bar{\sigma}_{2,e})_r(\gamma_1', \bar{z}^1; \gamma) = 0 .
 \end{aligned}$$

b) By using Proposition 15 and Proposition 16

$$\begin{aligned}
 & \partial_t (\bar{\sigma}_{2,2}^{-1}) \bar{\sigma}_{2,e}^{\epsilon, \epsilon, \epsilon} (x_1, x_2; \gamma) \quad \text{can be written as} \\
 & (\bar{\sigma}_{2,2}^{-1})_e(x_1, x_2; \gamma_1, \gamma_1') [a-b-c+d] \quad \text{with} \quad d = \bar{\sigma}_{2,e}^{\epsilon, \epsilon, \epsilon} (\gamma_0, \gamma_0'; \gamma) , \\
 & a = 8 \bar{\sigma}_{2,2}^{\epsilon, \epsilon, \epsilon} (\gamma_0, \gamma_0'; \gamma_1, \gamma_1') (\bar{\sigma}_{2,2}^{-1}) (\gamma_1, \gamma_1'; \gamma_2, \gamma_2') \bar{\sigma}_{2,2}^{\epsilon, \epsilon, \epsilon} (\gamma_2, \gamma_2'; \gamma_3, \gamma_3') \\
 &\quad \times (\bar{\sigma}_{2,2}^{-1}) (\gamma_3, \gamma_3'; \gamma_4, \gamma_4') (\bar{\sigma}_{2,e})_r(\gamma_4, \gamma_4'; \gamma) , \\
 & b = \bar{\sigma}_{2,2}^{\epsilon, \epsilon, \epsilon} (\gamma_0, \gamma_0'; \gamma_2, \gamma_2') (\bar{\sigma}_{2,2}^{-1})_r(\gamma_2, \gamma_2'; \gamma_4, \gamma_4') (\bar{\sigma}_{2,e})_r(\gamma_4, \gamma_4'; \gamma) , \\
 & c = 8 \bar{\sigma}_{2,2}^{\epsilon, \epsilon, \epsilon} (\gamma_0, \gamma_0'; \gamma_3, \gamma_3') (\bar{\sigma}_{2,2}^{-1}) (\gamma_3, \gamma_3'; \gamma_4, \gamma_4') (\bar{\sigma}_{2,e})_r(\gamma_4, \gamma_4'; \gamma) .
 \end{aligned}$$

The last three factors of a are by Proposition 16 a) equal to the last factor in c :

$$\begin{aligned}
 & 2 S_e(\gamma_2; \bar{z}^1) (\bar{\sigma}_{2,2})_r(\gamma_2', \bar{z}^1; \gamma_3, \gamma_3') (\bar{\sigma}_{2,2})_r(\gamma_3, \gamma_3'; \gamma_4, \gamma_4') (\bar{\sigma}_{2,e})_r(\gamma_4, \gamma_4'; \gamma) \\
 &= 2 S_e(\gamma_2; \bar{z}^1) (\bar{\sigma}_{2,e})_r(\gamma_2', \bar{z}^1; \gamma) .
 \end{aligned}$$

Thus $a = c$. Finally by Proposition 16 b) we have

$$\begin{aligned}
 & b = 8 (\bar{\sigma}_{2,2})_e(\gamma_0, \gamma_0'; \bar{z}_1^1, \bar{z}_2^1) (\bar{\sigma}_{2,2})_r(\bar{z}_1^1, \bar{z}_2^1; \gamma_3, \gamma_3') (\bar{\sigma}_{2,2})_r^{-1}(\gamma_3, \gamma_3'; \gamma_4, \gamma_4') \\
 &\quad \times (\bar{\sigma}_{2,e})_r(\gamma_4, \gamma_4'; \gamma) \\
 &= 8 (\bar{\sigma}_{2,2})_e(\gamma_0, \gamma_0'; \bar{z}_1^1, \bar{z}_2^1) (\bar{\sigma}_{2,e})_r(\bar{z}_1^1, \bar{z}_2^1; \gamma) = d .
 \end{aligned}$$

The same works of course for L.

□

Proposition 18 : $\partial_{\epsilon}^r \bar{G}_{k,e}^3(X; \overset{e}{Y}) = 0$ for $r = 0, 1, 2$.

Proof :

$r = 0, 1$: This follows immediately from Proposition 15 b)c).

$r = 2$: Recall that $\bar{G}_{k,e}^3 = \bar{G}_{k,e}^2 - \bar{G}_{k,2}^2 (\bar{G}_{2,2}^2)^{-1} \bar{G}_{2,e}^2$

$$\ddot{\bar{G}}_{k,e}^3 = \left[\ddot{\bar{G}}_{k,e}^2 - \ddot{\bar{G}}_{k,e}^2 ((\bar{G}_{2,2}^2)^{-1} \bar{G}_{2,e}^2) - 2 \dot{\bar{G}}_{k,2}^2 ((\bar{G}_{2,2}^2)^{-1} \bar{G}_{2,e}^2)^{\cdot} \right]$$

$$- \left[\bar{G}_{k,2}^2 ((\bar{G}_{2,2}^2)^{-1} \bar{G}_{2,e}^2)^{\cdot\cdot} \right]$$

$$= \bar{F}_{k,e} - \bar{G}_{k,2}^2 (\bar{G}_{2,2}^2)^{-1} \bar{F}_{2,e}, \text{ where}$$

$$\begin{aligned} \bar{F}_{k,e} &= \ddot{\bar{G}}_{k,e}^2 - \ddot{\bar{G}}_{k,2}^2 (\bar{G}_{2,2}^2)^{-1} \bar{G}_{2,e}^2 + 2 \dot{\bar{G}}_{k,2}^2 (\bar{G}_{2,2}^2)^{-1} \dot{\bar{G}}_{2,2}^2 (\bar{G}_{2,2}^2)^{-1} \bar{G}_{2,e}^2 \\ &\quad - 2 \dot{\bar{G}}_{k,2}^2 (\bar{G}_{2,2}^2)^{-1} \dot{\bar{G}}_{2,e}^2. \end{aligned}$$

Thus we can write

$$\ddot{\bar{G}}_{k,e}^3(X; \overset{e}{Y}) = \bar{F}_{k,e}(X; \overset{e}{Y}) - (\bar{G}_{k,2}^2 (\bar{G}_{2,2}^2)^{-1})_e(X; \overset{e}{y}_0, \overset{e}{y}'_0) \bar{F}_{2,e}(\overset{e}{y}_0, \overset{e}{y}'_0; \overset{e}{Y}).$$

But $\bar{F}_{2,e}(\overset{e}{y}_0, \overset{e}{y}'_0; \overset{e}{Y})$ is exactly the square bracket in the proof of Proposition 17 b) which vanishes, and analogously

$F_{k,e}^{e,r}(\underline{x};\underline{y}) = 0$. This proves the assertion. □

Proposition 19 : $k(\underline{x};\underline{y}) = \dot{k}(\underline{x};\underline{y}) = 0$

Proof :

Using (II.12) and Proposition 13 a)b), Proposition 15 a),

$$\begin{aligned} k(\underline{x};\underline{y}) &= G^{-1}(\underline{x};\underline{\gamma}) S(\underline{\gamma};\underline{y}) = 0 \\ \dot{k}(\underline{x};\underline{y}) &= G_e^{-1}(\underline{x};\underline{\gamma}_1) S(\underline{\gamma}_1; \underline{z}^1, \underline{z}^2; \underline{y}) \\ &\quad - G_e^{-1}(\underline{x};\underline{\gamma}_1) S(\underline{\gamma}_1; \underline{z}^1, \underline{z}^2; \underline{\gamma}_2) G_r^{-1}(\underline{\gamma}_2; \underline{\gamma}_3) S_r(\underline{\gamma}_3; \underline{y}) \\ &= (1-1) \cdot 2 G_e^{-1}(\underline{x};\underline{\gamma}) S_e(\underline{\gamma}; \underline{z}^1) S_r(\underline{z}^2; \underline{y}) = 0 \end{aligned}$$

□

Proposition 20 : a) $\partial_t^r K(\underline{x}_1, \underline{x}_2; \underline{y}_1, \underline{y}_2) = \partial_t^r K(\underline{x}_1, \underline{x}_2; \underline{y}_1, \underline{y}_2) = 0$
for $r = 0, 1$.

b) $\partial_t^r K(\underline{x}_1, \underline{x}_2; \underline{y}_1, \underline{y}_2) = 0$
for $r = 0, 1, 2$.

Proof :

Let C_ℓ^∞ denote the space of C^∞ functions on $\mathbb{R}^2$ vanishing with all its derivatives on $\{x : x^0 \in \mathbb{Z}\}$ and let L be the space of finite linear combinations $f = \sum_{i,j} f_i \otimes f_j$ with

$\{f_k\} \subset C_c^\infty$. Since $d_t^\alpha K_t : L_q^2 \rightarrow L_p^{2*}$ is bounded (see Proposition 12) and L is dense in L_q^2 it suffices to calculate the derivatives of

$$K_t f = \frac{1}{2} [(\bar{G}_{2,2}^2)^{-1} (\bar{G}_{1,1}^1)^{-1} \otimes (\bar{G}_{1,1}^1)^{-1}] C(t)^{-1} \otimes C(t)^{-1} f$$

for functions $f \in L$.

By definition $C(0)^{-1} f_1 = C(1)^{-1} f_1$ for $f_1 \in C^\infty$. Thus $C(t)C(0)^{-1} f_1 = f_1$ and

$$(III.7) \quad \dot{C}(t)C(0)^{-1} f_i = 0.$$

Furthermore $C(t)^{-1} f_1 = C(t)^{-1} C(t)C(0)^{-1} f_1 = C(0)^{-1} f_1$ and we can write

$$(III.8) \quad \partial_t^\alpha K_t f = \frac{1}{2} \partial_t^\alpha [(\bar{G}_{2,2}^2)^{-1} (\bar{G}_{1,1}^1)^{-1} \otimes (\bar{G}_{1,1}^1)^{-1}] \sum_{i,j} C(0)^{-1} f_i \otimes C(0)^{-1} f_j.$$

By using Propositions 15 and 16 and the factorization property

$$(\bar{G}_{2,2}^2)^{-1}(x_1, x_2; y_1, y_2) = \frac{1}{2} \bar{G}_e^{-1}(x_1, y_2) \bar{G}_r^{-1}(x_2, y_1)$$

we compute the necessary derivatives on the right hand side of

(III.8) :

$$\begin{aligned} 1) \quad & (\bar{G}_{2,2}^2)^{-1}(x_1, x_2; y_1, y_2) \\ &= \bar{G}_e^{-1}(x_1, y_0) \bar{G}_r^{-1}(x_2, y_0') 2S_e(y_0; \bar{z}^1)(\bar{G}_{2,2}^2)_r(\bar{z}^2, y_0'; y_1, y_2') \\ & \quad \times (\bar{G}_{2,2}^2)_r^{-1}(y_2, y_2'; y_1, y_2) \\ &= \bar{G}_e^{-1}(x_1; y_0) \bar{G}_r^{-1}(x_2; y_2) S_e(y_0, y_2) \dot{C}(y_1, y_2) \end{aligned}$$

$$\begin{aligned}
 2) \quad & (\tilde{G}_{2,2}^2)^{-1}(\overset{\ell}{x}_1, \overset{\ell}{x}_2; \overset{\ell}{y}_1, \overset{r}{y}_2) \\
 &= -2(\tilde{G}_{2,2}^2)_e^{-1}(\overset{\ell}{x}_1, \overset{\ell}{x}_2; \overset{\ell}{y}_0, \overset{\ell}{y}'_0) 2(\tilde{G}_{2,2}^2)_e(\overset{\ell}{y}_0, \overset{\ell}{y}'_0; \overset{\ell}{y}_1, \overset{r}{z}_1) \tilde{G}_r(\overset{r}{z}_1; \overset{\ell}{y}'_1) \\
 &\quad \times \frac{1}{2} \tilde{G}_e^{-1}(\overset{\ell}{y}_1; \overset{\ell}{y}_2) \tilde{G}_r^{-1}(\overset{\ell}{y}'_1; \overset{\ell}{y}_2) \\
 &= -(\tilde{G}_{2,2}^2)_e^{-1}(\overset{\ell}{x}_1, \overset{\ell}{x}_2; \overset{\ell}{y}_0, \overset{\ell}{y}'_0) (\tilde{G}_{2,2}^2)_e(\overset{\ell}{y}_0, \overset{\ell}{y}'_0; \overset{\ell}{y}_1, \overset{\ell}{y}) \tilde{G}_e^{-1}(\overset{\ell}{y}_1; \overset{\ell}{y}_2) \dot{C}(\overset{\ell}{y}, \overset{r}{y}_2) \\
 3) \quad & (\tilde{G}_{2,2}^2)^{-1}(\overset{\ell}{x}_1, \overset{\ell}{x}_2; \overset{\ell}{y}_1, \overset{r}{y}_2) \\
 &= 2(\tilde{G}_{2,2}^2)^{-1}(\overset{\ell}{x}_1, \overset{\ell}{x}_2; \overset{\ell}{y}_0, \overset{\ell}{y}'_0) 2S_e(\overset{\ell}{y}_0, \overset{\ell}{z}_1; (\tilde{G}_{2,2}^2)_r(\overset{\ell}{z}_1; \overset{\ell}{y}'_0; \overset{\ell}{y}_1, \overset{\ell}{y}'_1) \\
 &\quad \times (\tilde{G}_{2,2}^2)_r^{-1}(\overset{\ell}{y}_1, \overset{\ell}{y}'_1; \overset{\ell}{y}_2, \overset{\ell}{y}_2) \\
 &\quad - (\tilde{G}_{2,2}^2)_e^{-1}(\overset{\ell}{x}_1, \overset{\ell}{x}_2; \overset{\ell}{y}_0, \overset{\ell}{y}'_0) 8(\tilde{G}_{2,2}^2)_e(\overset{\ell}{y}_0, \overset{\ell}{y}'_0; \overset{\ell}{z}_1, \overset{\ell}{z}_2) (\tilde{G}_{2,2}^2)_r(\overset{\ell}{z}_1, \overset{\ell}{z}_2; \overset{\ell}{y}_1, \overset{\ell}{y}'_1) (\tilde{G}_{2,2}^2)_r^{-1}(\overset{\ell}{y}_1, \overset{\ell}{y}'_1; \overset{\ell}{y}_2, \overset{\ell}{y}_2) \\
 &= 2(\tilde{G}_{2,2}^2)(\overset{\ell}{x}_1, \overset{\ell}{x}_2; \overset{\ell}{y}_0, \overset{\ell}{y}_2) S_e(\overset{\ell}{y}_2; \overset{\ell}{y}) \dot{C}(\overset{\ell}{y}, \overset{\ell}{y}_2) \\
 &\quad - 2(\tilde{G}_{2,2}^2)_e^{-1}(\overset{\ell}{x}_1, \overset{\ell}{x}_2; \overset{\ell}{y}_0, \overset{\ell}{y}'_0) (\tilde{G}_{2,2}^2)_e(\overset{\ell}{y}_0, \overset{\ell}{y}'_0; \overset{\ell}{y}, \overset{\ell}{y}') \dot{C}(\overset{\ell}{y}, \overset{\ell}{y}_2) \dot{C}(\overset{\ell}{y}', \overset{\ell}{y}_2) .
 \end{aligned}$$

By (III.7) these terms do not contribute to $d_t^{\alpha} K_t|_{t=0}$. $\dot{C}(\overset{\ell}{y}, \overset{r}{y})$ can be replaced by $\dot{C}(\overset{\ell}{y}, \overset{\ell}{y})$ if f -and thus $C^{-1}f$ - has support on the right side of ℓ). The same is valid for the derivatives of $\sigma^{-1} \otimes \sigma^{-1} = (1 + kC) \otimes (1 + kC)$ by Proposition 19. Thus the assertions follows.

□

IV ANALYTICITY

We shall translate the results of the preceding section into decay properties of several kernels, or equivalently, into analyticity properties of their Fourier transforms.

Let us return to the notation of multiple derivatives $d_t^\alpha = \prod_{i \in I} \frac{d^{\alpha(i)}}{dt_i}$ described by a multiindex $\alpha = \{\alpha(i)\}_{i \in I}$. To each kernel $K(\underline{x}; \underline{y})$ we associate a multiindex-valued function $\beta[K, \underline{x}, \underline{y}]$. Furthermore let

$$d_\beta = \prod_k \prod_{\substack{\alpha_1 + \dots + \alpha_k = \beta \\ 0 \leq \alpha_j \leq 1}} \sum_{j=1}^k (d(\alpha_j) + 1) .$$

The list of kernels is as follows :

- a) $S^n(\underline{x}; \underline{y})$ for $n = 1, 2, 3$ and
 $\bar{S}^n(\underline{x}; \underline{y})$ for $n = 1, 2, 3$, with

$$\beta = \{\beta(i) = n \mid i_1 \leq i \leq i_2, \text{ whenever } x_i^\circ < i_1 < i_2 < y_i^\circ \neq i, j\} .$$

$$d_\beta = n |i_2 - i_1 + 1|$$

- b) $k(\underline{x}; \underline{y})$ with

$$\beta = \{\beta(i) = 2 \mid i_1 \leq i \leq i_2, \text{ whenever } x^\circ < i_1 < i_2 < y^\circ\} .$$

$$d_\beta = 2 |i_2 - i_1 - 1|$$

c) $K(x_1, x_2; y_1, y_2)$ with

$$\beta = \{ \beta(i) = 2 \}_{i_1 \leq i < i_2} \cup \{ \beta(i) = 3 \}_{i_2 \leq i \leq i_3} \cup \{ \beta(i) = 2 \}_{i_3 < i \leq i_4},$$

whenever $x_1^0 < i_1 < x_2^0 < i_2 < i_3 < y_1^0 < i_4 < y_2^0$

$$\begin{aligned} d_\beta &= 2(i_2 - i_1) + 3(i_3 - i_2) + 2(i_4 - i_3) \\ &\geq \frac{3}{2} |i_3 + i_4 - i_1 - i_2| + \frac{1}{2} |i_1 - i_2| + \frac{1}{2} |i_3 - i_4| \end{aligned}$$

d) $L(x_1, x_2; y)$ with

$$\beta = \{ \beta(i) = 2 \}_{i_1 \leq i < i_2} \cup \{ \beta(i) = 3 \}_{i_2 \leq i \leq i_3},$$

whenever $x_1^0 < i_1 < x_2^0 < i_2 < i_3 < y^0$.

$$d_\beta = 2(i_2 - i_1) + 3(i_3 - i_2) \geq \frac{3}{2} |2i_3 - i_1 - i_2| + \frac{1}{2} |i_1 - i_2|$$

Proposition 21 : For the kernels described above, one has with their associated β 's

$$d_{t_t}^{\alpha K_t(x; y)}|_{t_i = 0} = 0$$

if $\beta(i) \neq 0$ and $\alpha < \beta = \beta(K, \underline{x}, \underline{y})$, i.e. $\alpha(j) < \beta(j) \quad \forall j$.

This follows from the results of the preceding section and from the fact that t_i -derivatives commute. Next we will apply (I.4) to show the desired decay properties. If $A_{p,q}^{\sigma, \tau}$ denotes the class to which an operator K belongs according to Section II (for $S_{k,\ell}^n$ we can take $L_p^{\text{loc}}(\mathbb{R}^{2(k+\ell)})$, see Lemma 8 b), Lemma 1 and Lemma 2) then $K(\underline{x}; \underline{y}) = O(f(\underline{x}, \underline{y}))$ means that $f^{-1}(\underline{x}, \underline{y})K(\underline{x}, \underline{y})$ also defines an element in $A_{p,q}^{\sigma, \tau}$.

Theorem 22 :

Let $\varepsilon > 0$ be given. Then for λm_0^{-2} sufficiently small

- a) $S_{k,e}^n(X;Y) = O(e^{-nm_0(1-\varepsilon)d(X,Y)})$ for $n=1,2,3$
- b) $\bar{S}_{k,e}^n(X;Y) = O(e^{-nm_0(1-\varepsilon)d(X,Y)})$ for $n=1,2,3$
- c) $k(x,y) = O(e^{-2m_0(1-\varepsilon)|x^0-y^0|})$
- d) $L(x_1, x_2; y) = O(e^{-m_0(1-\varepsilon)[\frac{3}{2}|2y^0-x_1^0-x_2^0| + \frac{1}{2}|x_1^0-x_2^0|]})$
- e) $K(x_1, x_2; y_1, y_2) = O(e^{-m_0(1-\varepsilon)[\frac{3}{2}|y_1^0-y_2^0-x_1^0-x_2^0| + \frac{1}{2}|x_1^0-x_2^0| + \frac{1}{2}|y_1^0-y_2^0|]})$

where $d(x,y) = \min \{|x_i^0 - y_j^0| : x_i \in X, y_j \in Y\}$.

Proof :

Let again $K_{t,h}$ denote one of the kernels above. $K_{t,h}$ is analytic in h for $h \in \mathbb{R}$ (see the proof of Proposition 12) and all bounds for $K_{t,h}$ respectively for any of its component $\chi_{t,h} = (K_{t,h})_{\Pi_1, \Pi_2}$ are uniform in $t \in [0,1], I \in \mathbb{Z}, h \in \mathbb{R}$. In particular there are constants $C_1, C_2 > 0$ such that $|\chi_{t,h}|^{(C_1)} \leq C_2^{-1}$ for all t, I and h .

Let e denote functions of the form

$$e(\underline{x}, \underline{y}) = \chi_{\Delta_1}(\underline{x}) e^{C_1 d(i,j)} \chi_{\Delta_j}(\underline{y})$$

for varying i, j , and let $\beta(e)$ denote the value of

$\beta(K, \underline{x}, \underline{y})$ for $x \in \Delta_i, y \in \Delta_j$.

In order to prove the assertion we use (I. 4) to show property (I.12) for $\chi_{1,0}^f$ where

$$\chi_{t,h}^f(x,y) = \chi_{t,h}(x,y) f(x,y) \quad \text{and} \quad f(x,y) = e^{m_0(1-2\varepsilon)d_\beta(K,x,y)}.$$

$$|\chi_{1,0}^f|_{p,q}^{(c_1)} = \sup_{\varepsilon} \rho \|\chi_{1,0}^{e,f}\|_{p,q}$$

$$\leq \sup_{\varepsilon} \rho \left\| \int_0^1 \prod_{i=1}^k dt_i \sum_{\substack{\alpha_1 + \dots + \alpha_k = \beta(\varepsilon) \\ \alpha_j \neq 1}} \oint_{\partial R} \frac{k}{t} dh(\alpha_j) \frac{h(\alpha_j)^{-2}}{24i} \chi_{t,h}^{e,f} \right\|_{p,q}$$

$$\leq \sup_{\beta} e^{m_0(1-2\varepsilon)d_\beta} \left| \sum_{\substack{\alpha_1 + \dots + \alpha_k = \beta \\ \alpha_j \neq 1}} \oint_{\partial R} \frac{k}{t} dh(\alpha_j) \frac{h(\alpha_j)^{-2}}{24i} \right| \sup_{\substack{t \in (0,1) \\ h \in R}} \|\chi_{t,h}^e\|_{p,q}$$

$$\leq \sup_{\beta} \left[e^{m_0(1-2\varepsilon)d_\beta} \sum_{\substack{\alpha_1 + \dots + \alpha_k = \beta \\ 0 \neq \alpha_j \neq 1}} \frac{k}{t} e^{-m_0(1-\varepsilon)(d(\alpha_j)+1)} \right] \cdot c_2^{-1}$$

$$\leq c_3 c_2^{-1} < \infty$$

The third inequality follows for sufficiently large m_0 since the number of sets $\{\alpha_1, \dots, \alpha_k\}$ with $0 \neq \alpha_j \leq 1$, $\sum \alpha_j = \beta$ and $\sum (d(\alpha) + 1) = m$ is zero for $m < d_\beta$ and bounded by $(\text{const})^m$ for $m \geq d_\beta$. The square bracket is thus bounded by a finite constant C_3 . Finally 2ε may be replaced by ε

for λ_m^{-2} sufficiently small.

□

An equivalent formulation of Theorem 22 can be given in terms of the Fourier transformed kernels.

Let u, v, k, p, q be the momentum conjugate variables to x, y, τ, ξ, η , and let

$$\begin{aligned}
 (\text{IV.1}) \quad c(0, x) &\rightarrow C(u) = \frac{(2\pi)^{-1}}{u^2 + u^2} \\
 S(0, x) &\rightarrow S(\lambda; u) \\
 k(0, y) &\rightarrow (2\pi)^{-1} k(\lambda; v) \\
 L(-\frac{\xi}{2}, \frac{\xi}{2}; \tau) &\rightarrow (2\pi)^{-2} L(\lambda; k, p) \\
 K(-\frac{\xi}{2}, \frac{\xi}{2}; \tau - \frac{\eta}{2}, \tau + \frac{\eta}{2}) &\rightarrow (2\pi)^{-2} K(\lambda; k, p, q) \\
 S(-\frac{\xi}{2}; \gamma - \frac{\eta}{2}) S(\frac{\xi}{2}; \gamma + \frac{\eta}{2}) + S(-\frac{\xi}{2}; \gamma + \frac{\eta}{2}) S(\frac{\xi}{2}; \gamma - \frac{\eta}{2}) &\rightarrow R_0(\lambda; v, p, q) \\
 S_{2,2}^2(-\frac{\xi}{2}, \frac{\xi}{2}; \gamma - \frac{\eta}{2}, \gamma + \frac{\eta}{2}) &\rightarrow R(\lambda; v, p, q)
 \end{aligned}$$

denote corresponding Fourier transforms. Then

Corollary 23 : Under the same assumptions as in Theorem 22 the kernels of k, L and K as defined above are analytic and bounded by a constant in the region

$$||\omega k^1|, ||\omega p^1|, ||\omega q^1|, ||\omega v^1| \leq \frac{\omega_0 \varepsilon}{3} = \delta_1$$

$$||\omega p^0|, ||\omega q^0| \leq \frac{\omega_0}{2}(1-3\varepsilon) = \delta_0$$

$$||\omega v^0| \leq 2\omega_0(1-\varepsilon)$$

$$||\omega k^0| \leq 3\omega_0(1-\varepsilon) = 2(\omega_0 + \delta_0) \quad .$$

Proof :

Since our kernels are invariant by translation and decay exponentially in the difference variables it is clear that their Fourier transforms defined as in Definition 9 are bounded. Using the decay properties from Theorem 22 and the same for the x^1 -directions (a consequence of Euclidean invariance) the assertions follow by a simple generalization of the following argument. Let $K(x)dx$ be a bounded measure on $\mathbb{R}^2$ such that for every

$$j = (j^0, j^1) \in \mathbb{Z}^2$$

$$K_j(p) = \int \chi_{\Delta_j}(x) e^{ipx} K(x) dx$$

is defined having the properties

$$|K_j(p)| \leq e^{-d|j^0|} \text{ and } |K_j(p)| \leq e^{-d|j^1|}.$$

$$\text{Then } |K_j(p)| \leq |K_j(p)|^\alpha |K_j(p)|^{1-\alpha} \leq e^{-d(\alpha|j^0| + (1-\alpha)|j^1|)}$$

for $0 \leq \alpha \leq 1$, such that

$$K(p) = \int e^{ipx} K(x) dx = \sum_{j \in \mathbb{Z}^2} K_j(p) \text{ is bounded by const.}$$

$\sum_j \exp[-(d\alpha - |\text{Imp}^0|)|j^0| + (d(1-\alpha) - |\text{Imp}^1|)|j^1|]$. This sum converges for $|\text{Imp}^0| + |\text{Imp}^1| < d$ and $\alpha = |\text{Imp}^0|d^{-1}$.

□

V SOME PERTURBATION EXPANSIONS

Lemma 24 : For small $\lambda_0 \geq 0$ the functions $k(\lambda; v)$, $L(\lambda; k, p)$ and $K(\lambda; k, p, q)$ are C^∞ in λ for $\lambda \in [0, \lambda_0]$.

The derivatives are holomorphic and bounded (uniformly in λ) in the same region as shown in Corollary 23 for the original functions.

Proof :

Let $K_t(x; y)$ denote one of the kernels appearing in Theorem 22.

By [D] generalized Schwinger functions S_t are C^∞ in small

$\lambda \geq 0$. Thus using Proposition 11 the same follows for its t -

derivatives $d_t^\alpha S_t$. By the representation as convergent Neumann

series (Lemma 1 is also valid for $d_t^\alpha S_t$; this follows from

Proposition 12) $d_t^\alpha K_t$ is also C^∞ in λ .

Therefore $d_t^\alpha d_\lambda^n K_t = d_\lambda^n d_t^\alpha K_t$ so that Section IV can be repeated

for $d_\lambda^n K_t$. In the proof of Theorem 22 $K_{t,h}$ is replaced by

$(d_\lambda^n K_t)_h = D^n K_{t,h}$ with

$$(V.1) \quad D^n \langle Q_1, \dots, Q_m \rangle_{t,h} = (-1)^n \int d^2 y \langle Q_1, \dots, Q_m; :P(\phi):(\gamma_1); \dots; :P(\phi):(\gamma_n) \rangle_{t,h},$$

and D extended to a derivation.

At $h = 0$ (V.1) coincides with $d_\lambda^n \langle Q_1; \dots; Q_n \rangle_t$ [D].

Finally it is easily seen that Lemma 1 also applies for kernels

$D^n \langle Q_1; \dots; Q_u \rangle_{t,h}$ so that the bounds obtained in Section II for

$K_{t,h}$ remain valid for $D^n K_{t,h}$.

□

Proposition 25 : For small $\lambda \geq 0$ there are two C^∞ functions,
the physical mass $m(\lambda)$ with $m(0) = m_0$ and the field strength
 $z(\lambda)$ with $z(0) = 1$ such that

$$S(\lambda; p) = \frac{(2\pi)^{-1} z(\lambda)}{p^2 + m(\lambda)^2}$$

is analytic in p^2 for $\text{Re } p^2 > -M^2(\lambda)$ with $M(\lambda) \rightarrow 2m_0$ as
 $\lambda \rightarrow 0$ [GJS], [EEF].

The higher orders in λ of $m(\lambda)$ and $z(\lambda)$ can be computed
by using

$$(V.2) \quad \partial_\lambda^n z(\lambda) (m(\lambda)^2 - m_0^2)^s \Big|_{\lambda=0} = \partial_{p^2}^n F_s(p^2) \Big|_{p^2 = -m_0^2}$$

where

$$F_s(p^2) = \frac{1}{n!} \sum_{\substack{n_1 + \dots + n_k = n \\ n_j \geq 0}} (m_0^2 + p^2)^{n+s-k} \prod_{j=1}^k \partial_\lambda^{n_j} k(0; p) \quad .$$

Proof :

By Euclidean invariance $S(\lambda; p)$ only depends on p^2 . We may
 thus write $S(\lambda; z)$ instead of $S(\lambda; p(z))$ where $p(z) = (i\sqrt{z}, 0)$,
 and analogously for C and k .

Using Corollary 23 and Lemma 24 there is for each

$M^2 \in [3m_0^2, 4m_0^2)$ a positive λ_0 such that $k(\lambda; z)$ is analytic

in each $U_r = \{\operatorname{Re} z \leq M^2\} \cap \{|z - M^2| \leq \max(r, M)\}$, C^∞ and uniformly bounded by $O(\lambda)$ for $\lambda \in [0, \lambda_0]$. Thus by Rouché's theorem

$$S(\lambda; z)^{-1} = 2\pi \langle \omega_o^2 - z \rangle + k(\lambda; z))$$

has one simple zero in $\{\operatorname{Re} z \leq M^2\}$ (which is real since $\bar{S}(\lambda; \bar{z}) = S(\lambda; z)$), i.e.

$$S(\lambda; z)^{-1} = 2\pi \langle \omega^2(\lambda) - z \rangle \bar{z}(\lambda; z)^{-1}$$

with $z(\lambda; z)$ analytic and $\neq 0$ for $\operatorname{Re} z \leq M^2$. Using Lemma 24

$$(V.3) \quad \bar{z}(\lambda) = \bar{z}(\lambda; \omega^2(\lambda)) = -i \oint_{\partial U_o} S(\lambda; z) dz$$

$$\bar{z}(\lambda) \omega^2(\lambda) = -i \oint_{\partial U_o} z S(\lambda; z) dz$$

and therefore also $\bar{m}^2(\lambda)$ are C^∞ in λ . Formula (V.2) follows from

$$\partial_\lambda^n \bar{z}(\lambda) (\omega^2(\lambda) - \omega_o^2)^s = -i \oint_{\partial U_o} (z - \omega_o^2)^s \partial_\lambda^n S(\lambda; z) dz$$

and

$$\partial_\lambda^n S(\lambda; z) = \sum_{\substack{n_1 + \dots + n_k = n \\ n_i > 0}} (-2\pi)^k S(\lambda; z)^{k+1} \prod_{j=1}^k \partial_\lambda^{n_j} k(o; z) \quad .$$

□

Applying (V.2) for $s = 0, 1$ we obtain

$$(V.4) \quad \begin{aligned} \omega^2(o) &= \omega_o^2, & \bar{z}(o) &= 1, \\ \partial_\lambda \omega^2(o) &= \partial_\lambda k(o; i\omega_o, o), & \partial_\lambda \bar{z}(o) &= 0, \\ \partial_\lambda^2 \omega^2(o) &= \frac{1}{2} \partial_\lambda^2 k(o; i\omega_o, o), & \partial_\lambda^2 \bar{z}(o) &= \partial_p \partial_\lambda^2 k(o; i\omega_o, o). \end{aligned}$$

The perturbation expansion of $k(\lambda; p)$ begins with

$$(I, 5) \quad k(\lambda; p) = 2c_2 \lambda - \left[\sum_{j=1}^{2n-2} (j+2)(j+1) c_{j+2} c_j j! (2\pi) k_j(0) \right. \\ \left. + \sum_{j=2}^{2n-1} ((j+1) c_{j+1})^2 j! (2\pi) k_j(p) \right] \lambda^2 + O(\lambda^3)$$

where c_j is the coefficient of ϕ^j in the interaction polynomial $P(\phi)$.

$k_n(p)$ represents the graph  with total momentum p

$$k_n(p) = n^{-2} (2\pi)^{-(2n-1)} \int d^{2(n-1)} p \prod_{j=1}^{n-1} \left(\left(\frac{p}{u} + p_j'(p) \right)^2 + \omega_0^2 \right)^{-1}$$

$$p_j'(p) = -\frac{1}{u} \sum_{i=1}^{n-1} i p_i + \sum_{i=j}^{n-1} p_i$$

For example

$$(II, 6) \quad (2\pi) k_1(0) = \omega_0^{-2}$$

$$(2\pi)^2 k_2(0) = \frac{1}{2} (2\pi)^{-1} \omega_0^{-2}$$

$$k_2(i\chi, 0) = \frac{1}{4} (2\pi)^{-3} \int d^2 p \left(\left(i\frac{\chi}{2} + \frac{p_0}{2} \right)^2 + \frac{p_0^2}{4} + \omega_0^2 \right)^{-1} \left(\left(i\frac{\chi}{2} - \frac{p_0}{2} \right)^2 + \frac{p_0^2}{4} + \omega_0^2 \right)^{-1} \\ = 2(2\pi)^{-2} (4\omega_0^2 - \chi^2)^{-1/2} \chi^{-1} \arcsin \frac{\chi}{2\omega_0}$$

$$k_2(i\omega_0, 0) = \frac{\sqrt{3}}{18} (2\pi)^{-1} \omega_0^{-2}$$

Similar expansions as for $k(\lambda; p)$ can be found for

$K(\lambda; k, p, q)$ and $L(\lambda; k, p)$. In graphical notation (----- denote

amputated lines,  =  +  +  +

 =  +  + ... etc.)

$$k = \lambda \text{ ----- } - \lambda^2 \left[\text{-----} \text{ with a vertical line } + \text{-----} \text{ with a shaded oval} \right] + O(\lambda^3)$$

$$K = \lambda \text{ ----- } \times \text{-----}$$

$$- \lambda^2 \left[\text{-----} \text{ with a vertical line } + \left(\text{-----} \text{ with a shaded oval} + \text{-----} \text{ with a shaded oval} \right) + \text{-----} \text{ with a shaded oval} + \text{-----} \text{ with a shaded oval} \right] + O(\lambda^3)$$

$$L = \lambda \text{ ----- } \text{-----}$$

$$- \lambda^2 \left[\text{-----} \text{ with a vertical line } + \text{-----} \text{ with a shaded oval} + \text{-----} \text{ with a shaded oval} + \text{-----} \text{ with a shaded oval} \right] + O(\lambda^3)$$

In this graphical expansion numerical coefficients have been omitted in particular the C_i 's of the interaction polynomial.

Three terms which we will need later are

(V.7)

$$(2\pi)^2 \begin{array}{ccc} \frac{k}{2} + p & & \frac{k}{2} + q \\ & \searrow \quad \nearrow & \\ & \bullet & \\ & \nearrow \quad \searrow & \\ \frac{k}{2} - p & & \frac{k}{2} - q \end{array} = \alpha_1 = 6(2\pi)^{-1} c_4$$

$$2(2\pi)^2 \begin{array}{ccc} \frac{k}{2} + p & & \frac{k}{2} + q \\ & \searrow \quad \nearrow & \\ & \text{---} \bullet \text{---} & \\ & \nearrow \quad \searrow & \\ \frac{k}{2} - p & & \frac{k}{2} - q \end{array} = 18(2\pi)^{-1} ((p-q)^2 + m_0^2)^{-1} c_3^2$$

$$(2\pi)^2 \begin{array}{ccc} \frac{k}{2} + p & & \\ & \searrow \quad \nearrow & \\ & \bullet & \\ & \nearrow \quad \searrow & \\ \frac{k}{2} - p & & k \end{array} = \beta_1 = 3 c_3$$

VI THE POLES OF R AND S

In this section we analyze the kernels $S(\lambda; k)$ and $R(\lambda; h, p, q)$ at energies $k = (i\chi, 0)$ with χ in some neighbourhood of $2m(\lambda)$. For simplicity we denote them by $S(\lambda; \chi)$, $R(\lambda; \chi, p, q)$ respectively (and the same for R_0, k, L, K , etc.). As we shall see in the next section, R is not as directly related to physical quantities as for example the four point function R_{22} . But it is easier to handle because it satisfies the Bethe-Salpeter equation (see (II.11)).

$$(V.1) \quad R(\lambda; k, p, q) = R_0(\lambda; k, p, q) - \int d^2 p' d^2 q' R_0(\lambda; k, p, p') K(\lambda; k, p', q') R(\lambda; k, q', q)$$

which corresponds for fixed total momentum to the operator equation.

$$(V.2) \quad R(\lambda; k) = R_0(\lambda; k) - R_0(\lambda; k) K(\lambda; k) R(\lambda; k)$$

defined on the even subspace L_2^e of $L_2(\mathbb{R}^2)$. The formal solution of (VI.2) is

$$(V.3) \quad R(\lambda; k) = R_0(\lambda; k) (1 + K(\lambda; k) R_0(\lambda; k))^{-1}$$

which is similar to the formula (see (II.10))

$$(IV.4) \quad S(\lambda; k) = C(k) (1 + 2\pi k(\lambda; k) C(k))^{-1}.$$

In order to discuss the analyticity properties of S and R we have to look first at R_0 and K . The kernels $C(\chi)$ and $K(\lambda; \chi, p, q)$ are analytic in the region we are interested in.

Let A_δ be the Hardy space of symmetric functions analytic in $|\text{Im} p^0| < \delta_0 = \frac{m_0}{2} (1 - 3\varepsilon)$, $|\text{Im} p^1| < \delta_1 = \frac{1}{3} m_0 \varepsilon$, satisfying

$$\|f\|^2 := \sup_{|x_i| < \delta_i} \int d^2 p \left((p + i\alpha)^2 + 16\omega_0^2 \right)^{-2/3} |f(p + i\alpha)|^2 < \infty$$

and let A_δ denote its dual with respect to the product $\langle \cdot, \cdot \rangle = (\cdot, \cdot)_{L_2}$. An example of an element in A_δ is ε_0 defined by $\langle \varepsilon_0, f \rangle = f(0)$. From Corollary 23 it follows that $L(\lambda; \chi, \cdot)$ and $K(\lambda; \chi) \varepsilon_0$ are in A_δ depending analytically on χ for $|\text{Re} \chi| < 2(m_0 + \delta_0)$.

The analysis of [DE] is based on the decomposition

$$R_0(\lambda; \chi) = S_{10}(\lambda; \chi) + S_{20}(\lambda; \chi)$$

with ρ_{10}, ρ_{20} defined by

$$\begin{aligned}\langle f, S_{10}(\lambda, \chi) g \rangle &= \langle 1, Z(\lambda)^2 R_\lambda(\chi) f(c) g(c) \rangle \\ \langle f, S_{20}(\lambda, \chi) g \rangle &= \langle 1, Z(\lambda)^2 R_\lambda(\chi) [f g - \hat{f}(c) g(c)] \rangle \\ &\quad + \langle 1, [R_0(\lambda; \chi) - Z(\lambda)^2 R_\lambda(\chi)] f g \rangle\end{aligned}$$

where $R_\lambda(k)$ is constructed from $(k^2 + m^2(\lambda))^{-1}$ as $R_0(\lambda; k)$ is from $S(\lambda; k)$. Notice that the singularity of $R_\lambda(\chi)$ depends on λ while [DE] used a counterterm in the interaction to fix the mass at m_0 .

According to R_0 we split

$$T(\lambda; \chi) = K(\lambda; \chi) R_0(\lambda; \chi)$$

into $T_1 + T_2 = K\rho_{10} + K\rho_{20}$. ρ_{10} and T_1 can be given explicitly:

$$\begin{aligned}(V.5) \quad S_{10}(\lambda; \chi) &= r_0(\lambda; \chi) \varepsilon_0 \langle \varepsilon_0, \cdot \rangle \cdot \zeta(\chi)^{-1} \\ T_1(\lambda; \chi) &= r_0(\lambda; \chi) K(\lambda; \chi) \varepsilon_0 \langle \varepsilon_0, \cdot \rangle \cdot \zeta(\chi)^{-1} \\ \left(\int d^2 p d^2 q R_0(\lambda; \chi, p, q) &= r_0(\lambda; \chi) \zeta(\chi)^{-1} \right)\end{aligned}$$

$$\begin{aligned}(V.6) \quad r_0(\lambda; \chi) &= 4 Z(\lambda)^2 \arcsin\left(\frac{\chi}{2u(\lambda)}\right) \cdot \chi^{-1} \\ \zeta(\chi) &= (4u^2(\lambda) - \chi^2)^{-1/2}.\end{aligned}$$

The advantage of this decomposition is that the terms containing the singularities, ρ_{10} and T_1 , are rank one operators while ρ_{20} and T_2 can analytically be continued except for a branch point at $\chi = 2m(\lambda)$. More precisely let $\hat{}$ indicate a change of variables defined by

$$\hat{f}(\zeta(\chi)) = f(\chi) \quad \text{and let}$$

$$\hat{\mathcal{L}}_\lambda(\beta) = \{ \zeta = \pm \zeta(\chi) : \operatorname{Re} \zeta > -\beta, 0 \leq \operatorname{Re} \chi < 2(\omega_0 + \delta_0), \chi \notin [2m(\lambda), 2(\omega_0 + \delta_0)] \}.$$

Lemma 26 : For given small $\varepsilon > 0$ the operators

$$\hat{\mathcal{J}}_{10}(\lambda; \zeta), \hat{\mathcal{J}}_{20}(\lambda; \zeta) \in \mathcal{L}(A_\delta, A_\delta^*) \quad \text{and}$$

$$\hat{\mathcal{T}}_1(\lambda; \zeta), \hat{\mathcal{T}}_2(\lambda; \zeta) \in \mathcal{L}(A_\delta, A_\delta) \quad \text{are } C^\infty \text{ in } \lambda$$

for $0 < \lambda < \lambda(\varepsilon)$ and holomorphic in $\hat{\mathcal{D}}(\delta_1 - \varepsilon)$ together with their λ -derivatives.

The proof concerning the analyticity properties is given in [DE] while the C^∞ properties in λ follow by using Lemma 24.

An analogous splitting of R into $\rho_1 + \rho_2$ can be done as for R_0 . The only qualitative difference is that the singularity of ρ_1 does in general not lie at the branch point $\chi = 2m(\lambda)$.

Let us suppose that

$$a) \quad K(\lambda; k, p, q) = O(\lambda^n) \quad \text{and}$$

$$\alpha_n = (n!)^{-1} d_{\lambda}^n K(0; 2m_0, 0, 0) \neq 0$$

then

Lemma 27 : $R(\lambda; \chi) = \rho_1(\lambda; \chi) + \rho_2(\lambda; \chi)$ and for given small
 $\varepsilon > 0$ there is a unique C^∞ function $\zeta_1(\lambda)$ with values in
 $\hat{D}(\delta_1 - \varepsilon)$ such that
 $(\zeta - \zeta_1(\lambda)) \hat{\rho}_1(\lambda; \zeta), \hat{\rho}_2(\lambda; \zeta) \in L(A_\delta, A_\delta^*)$ are C^∞ in λ for
 $0 \leq \lambda \leq \lambda(\varepsilon)$ and holomorphic in $\hat{D}(\delta_1 - \varepsilon)$ together with
their λ -derivatives. Furthermore

$$(VII.7) \quad \hat{S}_1(\lambda; \zeta) = \hat{F}(\lambda; \zeta) (1 + \hat{T}_2(\lambda; \zeta)^*)^{-1} \varepsilon_0 \langle (1 + \hat{T}_2(\lambda; \zeta)^*)^{-1} \varepsilon_0, \cdot \rangle (\zeta - \zeta_1(\lambda))^{-1}$$

$$\hat{S}_2(\lambda; \zeta) = \hat{S}_{20}(\lambda; \zeta) (1 + \hat{T}_2(\lambda; \zeta))^{-1}$$

where $\zeta_1(\lambda)$ is the solution of

$$(VII.8) \quad \zeta_1(\lambda) + \hat{F}_0(\lambda, \zeta_1(\lambda)) \langle \varepsilon_0, (1 + \hat{T}_2(\lambda; \zeta_1(\lambda)))^{-1} \hat{K}(\lambda; \zeta_1(\lambda)) \varepsilon_0 \rangle = 0$$

and

$$\hat{F}(\lambda; \zeta) = \frac{(\zeta - \zeta_1(\lambda)) \hat{F}_0(\lambda; \zeta)}{\zeta + \hat{F}_0(\lambda, \zeta) \langle \varepsilon_0, (1 + \hat{T}_2(\lambda; \zeta))^{-1} \hat{K}(\lambda; \zeta) \varepsilon_0 \rangle}$$

Proof :

By taking $\lambda(\varepsilon)$ sufficiently small (not the same as in Lemma 26) $(1 + \hat{T}_2(\lambda; \zeta))^{-1}$ is analytic in $\hat{\mathcal{D}}(\delta_1 - \varepsilon)$ and C^∞ in $0 \leq \lambda \leq \lambda(\varepsilon)$ and $\hat{A}(\lambda; \zeta) = (1 + \hat{T}_2(\lambda; \zeta))^{-1} \hat{T}_1(\lambda; \zeta)$ is rank one and $O(\lambda \zeta^{-1})$. The trace of A is by (VI.5) equal to

$$\text{tr } A = \overline{F}_1 = \sum^{-1} \Gamma_o \langle \varepsilon_o, (1+T_2)^{-1} K \varepsilon_o \rangle .$$

Then applying the identity for rank one operators A with $\text{tr } A \neq -1$

$$(1+A)^{-1} = 1 - A(1 + \text{tr } A)^{-1}$$

we can write R as follows

$$\begin{aligned} R &= R_o(1+T_1+T_2)^{-1} = R_o(1+(1+T_2)^{-1}T_1)^{-1}(1+T_2)^{-1} \\ &= R_o\left(1 - \frac{(1+T_2)^{-1}T_1}{1+\overline{F}_1}\right)(1+T_2)^{-1} \\ &= (\mathcal{S}_{1o} + \mathcal{S}_{2o})(1+T_2)^{-1} - \frac{(\mathcal{S}_{1o} + \mathcal{S}_{2o})(1+T_2)^{-1}T_1(1+T_2)^{-1}}{1+\overline{F}_1} \\ &= \sum^{-1} \Gamma_o \varepsilon_o \langle \varepsilon_o, (1+T_2)^{-1} \cdot \rangle + \mathcal{S}_{2o}(1+T_2)^{-1} \\ &\quad - \frac{\sum^{-1} \Gamma_o \varepsilon_o \langle \varepsilon_o, (1+T_2)^{-1} K \varepsilon_o \rangle \sum^{-1} \Gamma_o \langle \varepsilon_o, (1+T_2)^{-1} \cdot \rangle}{1+\overline{F}_1} \end{aligned}$$

$$\begin{aligned}
 &= \frac{\rho_{20} (1+T_2)^{-1} K \varepsilon_0 \zeta^{-1} \Gamma_0 \langle \varepsilon_0, (1+T_2)^{-1} \cdot \rangle}{1 + \bar{F}_1} \\
 &= \frac{\zeta^{-1} \Gamma_0 \varepsilon_0 [1 + \bar{F}_1 \langle \varepsilon_0, (1+T_2)^{-1} K \varepsilon_0 \rangle \zeta^{-1} \Gamma_0] \langle \varepsilon_0, (1+T_2)^{-1} \cdot \rangle}{1 + \bar{F}_1} \\
 &= \frac{\rho_{20} (1+T_2)^{-1} K \varepsilon_0 \zeta^{-1} \Gamma_0 \langle \varepsilon_0, (1+T_2)^{-1} \cdot \rangle}{1 + \bar{F}_1} + \rho_{20} \\
 &= \frac{[1 - \rho_{20} (1+T_2)^{-1} K] \varepsilon_0 \Gamma_0 \langle \varepsilon_0, (1+T_2)^{-1} \cdot \rangle}{1 + \bar{F}_1} + \rho_{20} .
 \end{aligned}$$

Since $\hat{F}_1(\lambda; \zeta) = O(\lambda \zeta^{-1})$ the zeros of $(1 + \hat{F}_1(\lambda; \zeta))$ for small λ only can lie inside a small disc $N_d = \{\zeta : |\zeta| < d\}$ but not at $\zeta = 0$ (for $\lambda > 0$) by condition a). By using that $T_2^* = \rho_{20} K$ we obtain

$$(VI.9) \quad \rho_1 = R - \rho_2 = (1+T_2^*)^{-1} \varepsilon_0 \langle (1+T_2^*)^{-1} \varepsilon_0, \cdot \rangle \frac{\bar{\Gamma}_0}{\bar{\Gamma}_0 + \langle \varepsilon_0, (1+T_2)^{-1} K \varepsilon_0 \rangle}$$

which is C^∞ in λ and analytic for $\zeta \in \hat{D}(\delta_1 - \varepsilon)$ except at the zeros of the analytic function

$$\zeta + \hat{G}(\lambda; \zeta) = \zeta + \hat{\Gamma}_0(\lambda; \zeta) \langle \varepsilon_0, (1 + \hat{T}_2(\lambda; \zeta))^{-1} K(\lambda; \zeta) \varepsilon_0 \rangle .$$

But

$$(\zeta + \hat{Q}(\lambda; \zeta)) = [\zeta + \hat{Q}(\lambda; 0)] + [\hat{Q}(\lambda; \zeta) - \hat{Q}(\lambda; 0)]$$

with $[\zeta + \hat{Q}(\lambda; 0)]$ having a unique real zero $\zeta_{10}(\lambda)$ and

$$(VI.4c) \quad |\zeta + \hat{Q}(\lambda; \zeta)| > \frac{1}{2}|\zeta| > |\hat{Q}(\lambda; \zeta) - \hat{Q}(\lambda; 0)|$$

for small λ (determines $\lambda(\epsilon)$) and $\zeta \in \partial N_d$. So by Rouché's theorem $\zeta + \hat{Q}(\lambda; \zeta)$ has a unique simple zero $\zeta_1(\lambda)$ in N_d and thus in $\hat{V}(\delta_1 - \epsilon)$.

Obviously $\zeta_1(\lambda)$ is real, C^∞ in λ (see also the proof of Theorem 33) and $\zeta_1(0) = 0$. Therefore

$$\frac{\hat{f}_0(\lambda; \zeta)}{\zeta + \hat{Q}(\lambda; \zeta)} = (\zeta - \zeta_1(\lambda))^{-1} \hat{r}(\lambda; \zeta)$$

for some analytic function $\hat{r}(\lambda; \zeta)$ and ρ_1 takes the desired form

$$\rho_1 = r(1 + T_2^*)^{-1} \mathcal{E}_0 \langle (1 + T_2^*)^{-1} \mathcal{E}_0, \cdot \rangle (\zeta - \zeta_1)^{-1}.$$

□

Next we shall look at the poles of $\hat{S}(\lambda; \zeta)$ in $\hat{V}(\delta_1 - \epsilon)$. To do so it suffices by (VI.4) to know $\hat{k}(\lambda; \zeta)$ in this region.

Proposition 28 :

$$k(x; y) = -\lambda^2 \langle : \mathcal{P}'(\phi) : (x) P_2 : \mathcal{P}'(\phi) : (y) \rangle \\ - \lambda^2 \langle : \mathcal{P}'(\phi) : (x) [1 - P_0 - P_1 - P_2] : \mathcal{P}'(\phi) : (y) \rangle + \lambda \langle : \mathcal{P}''(\phi) : (x) \rangle \delta(x-y)$$

$$\hat{k}(\lambda; \zeta) = -(2\bar{u})^{-1} \langle \hat{L}(\lambda; \zeta, \cdot), \hat{R}(\lambda; \zeta) \hat{L}(\lambda; \zeta, \cdot) \rangle \\ - 2\bar{u} \hat{D}_2(\lambda; \zeta) + 2\bar{u} D_4(\lambda)$$

where $\hat{D}_2(\lambda; \zeta)$ is analytic in $\hat{D}(\delta_1 - \epsilon)$.

Proof :

With the short notation

$$\langle \mathcal{P}'' \rangle (x; y) = \langle : \mathcal{P}''(\phi) : (x) \rangle \delta(x-y)$$

$$\langle \mathcal{P}' ; \phi \rangle (x; y) = \langle : \mathcal{P}'(\phi) : (x) ; \phi(y) \rangle , \quad \text{etc.}$$

We can use integration by parts to write

$$\sigma_{1,1}^1 = 1 - \lambda \langle \mathcal{P}' ; \phi \rangle = 1 + (\lambda^2 \langle \mathcal{P}' ; \mathcal{P}' \rangle - \lambda \langle \mathcal{P}'' \rangle) \zeta \\ = 1 + \zeta (\lambda^2 \langle \mathcal{P}' ; \mathcal{P}' \rangle - \lambda \langle \mathcal{P}'' \rangle) = 1 - \lambda \langle \phi ; \mathcal{P}' \rangle$$

and therefore (writing $\sigma_{1,1}^1 = \sigma$)

$$k \zeta = \sigma^{-1} - 1 = (1 - \sigma) \sigma^{-1} (1 - \sigma) + (1 - \sigma) \\ = \lambda^2 \langle \mathcal{P}' ; \phi \rangle \sigma^{-1} \langle \mathcal{P}' ; \phi \rangle + \lambda \langle \mathcal{P}' ; \phi \rangle \\ = \lambda^2 \langle \mathcal{P}' ; \phi \rangle \sigma^{-1} \zeta^{-1} \langle \phi ; \mathcal{P}' \rangle \zeta - (\lambda^2 \langle \mathcal{P}' ; \mathcal{P}' \rangle - \lambda \langle \mathcal{P}'' \rangle) \zeta$$

By using the projections P_n (see Definitions 6 and 7 and (II.3)) k can thus be written as

$$\begin{aligned} k &= \lambda^2 \langle \phi' P_1 \phi' \rangle - \lambda^2 \langle \phi' (1 - P_0) \phi' \rangle + \lambda \langle \phi'' \rangle \\ &= -\lambda^2 \langle \phi' P_2 \phi' \rangle - \lambda^2 \langle \phi' (1 - P - P - P_2) \phi' \rangle + \lambda \langle \phi' \rangle . \end{aligned}$$

As a consequence of Theorem 22 a)

$$D_2(\tau) := \lambda^2 \langle : \phi'(\phi) : (0) [1 - P_0 - P_1 - P_2] : \phi'(\phi) : (\tau) \rangle = \lambda^2 O(e^{-3m_0(1-\epsilon)|\tau|})$$

(with respect to the same $A_{p,q}^{*,*}$ as k) such that its Fourier transform $D_2(\lambda; k)$ is analytic for

$$|\operatorname{Im} k^0| + |\operatorname{Im} k^1| < 3m_0(1 - \epsilon). \text{ By defining}$$

$L^*(y; x_1, x_2) = L(x_1, x_2; y)$ (see Proposition 10) it is easily seen that

$$B_1(\tau) := \lambda^2 \langle : \phi'(\phi) : (0) P_2 : \phi'(\phi) : (\tau) \rangle = (L^* S_{z,z}^2 L)(0, \tau) ,$$

$$(VII 11) \quad B_1(\lambda; \chi) = (2\pi)^{-2} \langle L(\lambda; \chi, \cdot), R(\lambda; \chi) L(\lambda; \chi, \cdot) \rangle .$$

In terms of B_1, D_2 and $D_1(\lambda) = \lambda \langle : P''(\phi) : (0) \rangle$ k takes the form

$$k(\lambda; \chi) = -2\pi B_1(\lambda; \chi) - 2\pi D_2(\lambda; \chi) + 2\pi D_1(\lambda)$$

□

From this it follows that in an even theory (where $L \equiv 0$) the two point function is analytic up to $|\operatorname{Re} \chi| < 3m_0(1 - \epsilon)$, except for a pole of $\chi = m(\lambda)$. But if the interaction contains odd terms then the situation is changed as we shall see in the next lemma. Let

- a) $K(\lambda; \chi, p, q) = \mathcal{O}(\lambda^n)$ and $\alpha_n := (n!)^{-2} \partial_\lambda^n K(0, 2m_0, 0, 0) \neq 0$
- b) $L(\lambda; \chi, p) \equiv 0$ or $L(\lambda; \chi, p) = \mathcal{O}(\lambda^m)$ and $\beta_m := (m!)^{-1} \partial_\lambda^m L(0; 2m_0, 0) \neq 0$
- c) $\gamma := -\frac{1}{3}(2\pi)^{-1} m_0^{-2} \beta_m^2 \alpha_n^{-1} \neq 0$ if $n=2m$.

Lemma 29 : Under the assumption a), b) with $L \not\equiv 0$ and c), and for given small $\epsilon > 0$ there is a C^∞ function $\zeta_2(\lambda)$ with values in $\hat{D}(\delta_1 - \epsilon)$ and $\zeta_2(0) = 0$ such that $\hat{S}(\lambda; \zeta)$ is C^∞ in $0 \leq \lambda \leq \lambda(\epsilon)$ and holomorphic in ζ for $\zeta \in \hat{D}(\delta_1 - \epsilon)$ together with its λ -derivatives, except for simple poles at $\sqrt{3}m(\lambda)$ and $\zeta_2(\lambda)$. Furthermore $\hat{S}(\lambda; \zeta)$ has a zero at $\zeta_1(\lambda)$.

Proof :

As in the previous proof we write $k = -2\Pi B_1 - 2\Pi D_2 + 2\Pi D_1$.

Then according to Lemma 27 we split B_1 into $B + D_3$ where

R (see (VI.11)) is replaced by ρ_1 and ρ_2 respectively. Then

the condition for $S(\lambda; \chi)$ to have a pole is

$$1 + 2\bar{u}k(\lambda; \chi)C(\chi) = 0$$

and can be rewritten as

$$(VII.12) \quad (1 - (2\bar{u})^2(1 + D(\lambda; \chi))B(\lambda; \chi)C(\chi))(1 + D(\lambda; \chi)) = 0$$

$$\text{where } D = (1 - (2\pi)^2(D_3 + D_2 - D_1)C)^{-1} - 1.$$

Repeating the proof of Lemma 24 for D_2 we obtain that $\hat{D}_3(\lambda; \zeta) + \hat{D}_2(\lambda; \zeta) - D_1(\lambda)$ is C^∞ in λ and analytic and uniformly bounded by $O(\lambda)$ in $\hat{D}(\delta_1 - \epsilon)$ and thus the same is true for $\hat{D}(\lambda; \zeta)$. Notice that by Proposition 25 we can exclude a region $M = \{\zeta(\chi) \in \hat{D}(\delta_1 - \epsilon) : |\operatorname{Re} \chi| < \frac{3}{2}m_0\}$ from our consideration. So there remains to find the zeros in $\hat{D}(\delta_1 - \epsilon) \setminus (M \cup \{\zeta_1(\lambda)\})$ of the function

$$\hat{F}_2(\lambda; \zeta) - 1 = (2\bar{u})^2(1 + \hat{D}(\lambda; \zeta))\hat{B}(\lambda; \zeta)\hat{C}(\lambda; \zeta) - 1$$

which is by Lemma 27 analytic there and has by (VI.9), (VI.11) the representation

$$(VII.13) \quad \hat{F}_2(\lambda; \zeta) - 1 = (2\bar{u})^2(1 + \hat{D}(\lambda; \zeta))\hat{C}(\lambda; \zeta) \frac{\lambda^{2m} \hat{t}(\lambda; \zeta)}{\zeta - \lambda^n \hat{q}(\lambda; \zeta)} - 1$$

with

$$\begin{aligned} t(\lambda; \chi) &= \lambda^{-2m} (2\bar{u})^{-2} \langle \varepsilon_0, (1 + T_2(\lambda; \chi))^{-1} L(\lambda; \chi, \cdot) \rangle^2 \Gamma_0(\lambda; \chi) \\ q(\lambda; \chi) &= -\lambda^{-n} \langle \varepsilon_0, (1 + T_2(\lambda; \chi))^{-1} K(\lambda; \chi) \varepsilon_0 \rangle \Gamma_0(\lambda; \chi) \end{aligned}$$

Now let $Q(\lambda) = \hat{q}(\lambda; \zeta_1(\lambda)) = \lambda^{-n} \zeta_1(\lambda)$,

$$\delta(\lambda; \zeta) = -(2q)^2 (1 + \hat{D}(\lambda; \zeta)) \hat{C}(\lambda; \zeta) \hat{E}(\lambda; \zeta) Q(\lambda)^{-1}$$

and $\gamma(\lambda) = \gamma(\lambda; \zeta_1(\lambda))$. Therefore $\gamma(0) = \gamma(0, 0) = \gamma$. Then

using the assumptions a)b)c) it is possible to choose a complex neighbourhood N of zero and a $\lambda' > 0$ such that for $\lambda \in [0, \lambda']$,

$\zeta \in N$ and $\zeta' \in \hat{D}(\delta_1 - \epsilon) \setminus (N \cup M)$

I) $\hat{C}(\lambda; \zeta)$ and $\hat{q}(\lambda; \zeta)$ are bounded away from zero

II) $\hat{F}_2(\lambda; \zeta') - 1 \neq 0$

III) $|Q(\lambda)(1 - \lambda^{2m-n} \gamma(\lambda))| > |q(\lambda; \zeta) - Q(\lambda) - \lambda^{2m-n}(\gamma(\lambda; \zeta) - \gamma(\lambda))Q(\lambda)|$

(will be used for Proposition 30)

From I) and II) it follows that the zeros of $\hat{F}_2(\lambda; \zeta) - 1$ are identical with the solutions of

$$\begin{aligned} (\text{VI.14}) \quad Q(\lambda)(1 - \lambda^{2m-n} \delta(\lambda)) - \lambda^{-n} \zeta \\ + [\hat{q}(\lambda; \zeta) - Q(\lambda) - \lambda^{2m-n}(\delta(\lambda; \zeta) - \delta(\lambda))Q(\lambda)] = 0. \end{aligned}$$

But $Q(\lambda)(1 - \lambda^{2m-n} \gamma(\lambda)) - \lambda^{-n} \zeta = 0$ has exactly one solution

$\zeta_{20}(\lambda) = \lambda^n Q(\lambda)(1 - \lambda^{2m-n} \gamma(\lambda))$ in N for sufficiently small

$\lambda > 0$. Moreover since

$$\begin{aligned} (\text{VI.15}) \quad & |Q(\lambda)(1 - \lambda^{2m-n} \delta(\lambda)) - \lambda^{-n} \zeta| \\ & > |\hat{q}(\lambda; \zeta) - Q(\lambda) - \lambda^{2m-n}(\delta(\lambda; \zeta) - \delta(\lambda))Q(\lambda)| \end{aligned}$$

on ∂N Rouché's theorem guarantees also a unique solution

$\zeta_2(\lambda)$ of (VI.14) in N . For obvious reasons $\zeta_2(\lambda)$ is real,

C^∞ in λ (see also the proof of Theorem 33), $\zeta_2(0) = 0$ and $\zeta_1(\lambda)$

is a zero of $\hat{S}(\lambda; \zeta)$.

Next we give criteria which allow to decide whether the singularity $\chi_1(\lambda) = \sqrt{4m^2(\lambda) - \zeta_1(\lambda)^2}$ of $R(\lambda; \chi)$ and $\chi_2(\lambda) = \sqrt{4m^2(\lambda) - \zeta_2(\lambda)^2}$ of $S(\lambda; \chi)$ lies on the first sheet (i.e. $\chi(\zeta > 0)$) of the manifold $\mathcal{D}(\delta_1) = \{\chi(\zeta) = \sqrt{4m^2(\lambda) - \zeta^2} : \zeta \in \hat{\mathcal{D}}(\delta_1)\}$ or on its second sheet (i.e. $\chi(\zeta < 0)$).

Proposition 30 : For $\lambda > 0$ sufficiently small we have

- a) $\zeta_1(\lambda) > 0$ if $\alpha_n < 0$.
- b) $\zeta_1(\lambda) < 0$ if $\alpha_n > 0$.
- c) $\zeta_2(\lambda) > 0$ if $\alpha_n < 0$ and either $n < 2m$
or $n = 2m$ with $0 \leq \delta < 1$.
- d) $\zeta_2(\lambda) < 0$ if $\alpha_n > 0$ or
 $[\alpha_n < 0, m = 2n \text{ and } \delta > 1]$

Proof :

a) b) See the proof of Lemma 27 :

For small $\lambda > 0$ the signs of α_n and $\hat{Q}(\lambda; 0)$ and thus of $-\zeta_{10}(\lambda)$ are identical. This remains true for $\zeta_{10}(\lambda)$ replaced by $\zeta_1(\lambda)$ since (VI.10) is also satisfied for $\zeta \in \partial N_d^\pm$ where $N_d^\pm = \{\zeta \in \mathbb{C} : |\zeta| \leq r, \pm \operatorname{Re} \zeta \leq 0\}$.

c) d) See the proof of Lemma 29 :

$\zeta_2(\lambda)$ has the same sign as $\zeta_{20}(\lambda)$ since by III the condition

(VI.15) is also satisfied on the axis $\text{Re } \zeta = 0$. By using this and the fact that for sufficiently small $\lambda > 0$ $Q(\lambda)$ and $\gamma(\lambda)$ have the same sign as $-\alpha_n$ we can look at $\zeta_{20}(\lambda) = \lambda^{n_Q(\lambda)} (1 - \lambda^{2m - n_\gamma(\lambda)})$ to conclude the assertions. □

V II THE CONNECTION BETWEEN THE MASS SPECTRUM AND THE POLES OF S AND R_{22}

In this section we will more or less follow the analysis of Spencer, Zirilli [SZ] in order to outline the connection between the poles of $\hat{S}(\lambda; \zeta)$ and $\hat{R}_{22}(\lambda; \zeta)$ (see Definition 11) and the spectrum of the mass operator $M = (H^2 - P^2)^{\frac{1}{2}}$ acting on the physical Hilbert space $\mathcal{H}$. Since we also include odd powers in the interaction polynomial P the three point function R_{21} does generally not vanish and the four point function R_{22} does not coincide with the one particle irreducible four point function R .

Definition 11 :

To the list (IV.1) we add two more kernels respectively their Fourier transforms

$$\begin{aligned} C_2 G_{21}^1(-\frac{x}{2}, \frac{x}{2}; \tau) &\rightarrow R_{21}(\lambda; k, p) \\ C_2 G_{22}^1(-\frac{x}{2}, \frac{x}{2}; \tau - \frac{x}{2}, \tau + \frac{x}{2}) &\rightarrow R_{22}(\lambda; k, p, q) \end{aligned}$$

Let $\Omega \in H$ denote the vacuum, E_0 the orthogonal projection onto it and let $\langle \phi \rangle = (\Omega, \phi(0)\Omega)$. Furthermore for open rectangles $N = N_m \times N^1$ contained in $M = (0, 2m(\lambda)) \times (-\Delta, \Delta)$ let $E(N)$ denote the orthogonal projection in H onto the subspace of states with energy-momentum $k = (k^0, k^1)$ for which $((k^0)^2 - (k^1)^2)^{\frac{1}{2}}, k^1 \in N$. Δ is for the moment arbitrary.

Then considering vectors

$$(VII.1) \quad \theta(t, f)\Omega = e^{t^0 H + i t^1 P} \theta(0, f)\Omega$$

(smeared out in t^1 with functions in $C_0^\infty(\mathbb{R})$) where

$$\begin{aligned} \theta(0, f) &= \theta_1(f_1) + \theta_2(f_2) \\ &= E(M) f_1 \hat{\psi}(0) + E(M) \int dx f_2(x) (1 - E_0) \hat{\psi}(0, -\frac{x}{2}) \hat{\psi}(0, \frac{x}{2}) \end{aligned}$$

$$f = (f_1, f_2) \in \mathbb{R} \times C_0^\infty(\mathbb{R}) \quad , \quad \hat{\psi} = \psi - \langle \psi \rangle$$

and using that

$$\begin{aligned} (VII.2) \quad & (\psi(x_1) \dots \psi(x_k) \Omega, (1 - E_0) e^{-t^0 H + i t^1 P} \psi(y_1) \dots \psi(y_e) \Omega) \\ &= S(x_1, \dots, x_k; y_1 + t, \dots, y_e + t) \end{aligned}$$

for $t^0 > 0$ and $x_k^0 < x_{k-1}^0 < \dots < x_1^0 < y_1^0 < \dots < y_e^0$,

we obtain the following relation

$$(VII.3) \quad (\theta(t, f) \mathcal{R}, E(N) \theta(s, g) \mathcal{R})$$

$$= \frac{1}{2\pi i} \int_{N^1} dk^1 \oint_{\Gamma} dk^0 (\theta(t, f) \mathcal{R}, (H-k^0)^{-1} \delta(P-k^1) \theta(s, g) \mathcal{R})$$

$$= \frac{1}{2\pi i} \int_{N^1} dk^1 \oint_{\Gamma} dk^0 \int_{-\infty}^{\infty} dx^1 \int_0^{\infty} dx^0 (\theta(t, f) \mathcal{R}, e^{-x^0(H-k^0)} e^{ix^1(P-k^1)} \theta(s, g) \mathcal{R})$$

$$= \frac{1}{2\pi i} \int_{N^1} dk^1 \oint_{\Gamma} dk^0 e^{(s^0+t^0)k^0 + i(s^1-t^1)k^1} \\ \times \int_{-\infty}^{\infty} dx^1 \int_0^{\infty} dx^0 (\theta(t, f) \mathcal{R}, e^{-x^0(H-k^0)} e^{ix^1(P-k^1)} \theta(s, g) \mathcal{R})$$

$$+ \frac{1}{2\pi i} \int_{N^1} dk^1 \oint_{\Gamma} dk^0 e^{i(s^1-t^1)k^1 + (s^0+t^0)k^0} \int_{-\infty}^{\infty} dx^1 \int_{-s^0-t^0}^0 dx^0 (\theta(t, f) \mathcal{R}, e^{-x^0(H-k^0)} e^{ix^1(P-k^1)} \theta(s, g) \mathcal{R})$$

$$= \frac{1}{2\pi i} \int_{N^1} dk^1 \oint_{\Gamma} dk^0 e^{i(s^1-t^1)k^1 + (s^0+t^0)k^0} \int dx^1 e^{x^0 k^0 - ix^1 k^1} (\theta(t, f) \mathcal{R}, e^{-ix^0 H + ix^1 P} \theta(s, g) \mathcal{R})$$

$$= \frac{1}{2\pi i} \int_{N^1} dk^1 \oint_{\Gamma} dk^0 e^{(s^0+t^0)k^0 + i(s^1-t^1)k^1} [\langle 1 \otimes \tilde{f}_z, R_{22}(\lambda; (ik^0, k^1)) 1 \otimes \tilde{g}_z \rangle$$

$$+ f_z \langle R_{21}(\lambda; (ik^0, k^1)), 1 \otimes \tilde{g}_z \rangle + \langle R_{21}(\lambda; (ik^0, k^1)), 1 \otimes \tilde{f}_z \rangle g_1$$

$$+ f_1 S(\lambda; (ik^0, k^1)) g_1]$$

Γ is the boundary of a sufficiently small complex neighbourhood of a sufficiently large closed interval contained in the set $\{(\mu^2 + (k^1)^2)^{\frac{1}{2}} : \mu \in N_m\}$.

We have also used that

$$\int dx^1 \int_{-\infty}^0 dx^0 e^{x^0 k^0 - i x^1 k^1} (\theta(0, f) \mathcal{R}, e^{-i x^0 H + i x^1 P} \theta(0, g) \mathcal{R})$$

is analytic for $\text{Re } k^0 > 0$, that

$$\int dx^1 \int_{-s^0 - t^0}^0 dx^0 (\theta(0, f) \mathcal{R}, e^{-x^0 (H - k^0) + i x^1 (P - k^1)} \theta(0, g) \mathcal{R})$$

is an entire function of k^0 , and that $R_{21}(\lambda; \chi)$ and $R_{22}(\lambda; \chi)$ are analytic in $\mathcal{D}(\delta_1 - \varepsilon) \setminus \{\chi_1(\lambda), \chi_2(\lambda)\}$. The latter statement follows from

Proposition 31 : a) $R_{21}(\lambda; k, \cdot) = R(\lambda; k) L(\lambda; k, \cdot) S(\lambda; k)$

b) $R_{22}(\lambda; k) = R(\lambda; k)$

$$+ R(\lambda; k) L(\lambda; k, \cdot) S(\lambda; k) \langle R(\lambda; k) L(\lambda; k, \cdot), \cdot \rangle$$

Proof :

a) Using the definitions of $k, \sigma_{k, \lambda P}^n$ and L (see Definitions 8, 9 and Proposition 10) we can write

$$\mathcal{G}_{2,1}^1 = \mathcal{G}_{2, \lambda P'}^1 \mathcal{C} = \mathcal{G}_{2, \lambda P'}^2 \mathcal{C} - \mathcal{G}_{2,1}^1 (\mathcal{G}_{1,1}^1)^{-1} \mathcal{G}_{1, \lambda P'}^1 \mathcal{C} = \mathcal{G}_{2, \lambda P'}^2 \mathcal{C} - \mathcal{G}_{2,1}^1 k \mathcal{C}.$$

Thus

$$\begin{aligned}
 (\text{VII.4}) \quad C_2 G_{2,1}^1 &= C_2 G_{2,1P}^2 (1+kC)^{-1} = C_2 G_{2,1P}^2 S_{1,1}^1 \\
 &= S_{2,2}^2 L S_{1,1}^1
 \end{aligned}$$

and the corresponding equation in p-space has the desired form.

b) The analogue of (V II.4) for $C\sigma_{1,2}^1$ leads to

$$G_{1,2}^1 = G_{1,1}^1 L^* S_{2,2}^2.$$

By inserting this into the definition of $S_{2,2}^2$,

$$C_2 G_{2,2}^1 = C_2 G_{2,1}^1 (G_{1,1}^1)^{-1} G_{1,2}^1 + S_{2,2}^2,$$

and using (V II.4) one obtains

$$(\text{VII.5}) \quad C_2 G_{2,2}^1 = S_{2,2}^2 + S_{2,2}^2 L S_{1,1}^1 L^* S_{2,2}^2.$$

□

Now with the usual definition of a particle as an irreducible representation of the Poincaré group we get

Proposition 32 : Under the assumptions a)b)c) of Section VI the following is true; in a $\lambda P(\phi)_2$ theory with small λ there are at most two particles with mass in the interval $(0, 2m(\lambda))$.

Proof :

It can easily be checked that (V II.3) is valid for vectors of the form

$$\begin{aligned} \Theta(0, h_1, h_2) \Omega &= h_1 E(H) \hat{\psi}(0) \Omega \\ &+ E(H) \int d^2 x \, h_2(x) (1 - E_0) \hat{\psi}(x, -\frac{x^1}{2}) \hat{\psi}(x, \frac{x^1}{2}) \end{aligned}$$

not only when k_2 is of the form $\delta \otimes f_2$ but also for $h_2 = (\delta \otimes f_2) \circ \rho$ where ρ is a Euclidean rotation in two dimensions. More precisely

$$\begin{aligned} (\text{VII.6}) \quad & (\Theta(0, f_1, (\delta \otimes f_2) \circ S) \Omega, E(N) \Theta(0, g_1, (\delta \otimes g_2) \circ S) \Omega) \\ &= \frac{1}{2\pi i} \int_{\Gamma} dk^1 \oint dk^0 \left[\langle (1 \otimes \tilde{f}_2) \circ S, R_{22}(\lambda; (ik^0, k^1)) ((1 \otimes \tilde{g}_2) \circ S) \rangle \right. \\ &\quad \left. + \int_1 \langle R_{21}(\lambda; (ik^0, k^1)), (1 \otimes \tilde{g}_2) \circ S \rangle \right. \\ &\quad \left. + \langle R_{21}(\lambda; (ik^0, k^1)), (1 \otimes \tilde{f}_2) \circ S \rangle g_1 + \int_1 S(\lambda; (ik^0, k^1)) g_1 \right]. \end{aligned}$$

Additional terms arising when the support of $(\delta \otimes f_2) \circ \rho^{-1}$ and the x^0 -translates of $(\delta \otimes g_2) \circ \rho$ overlap give rise to a function of x^0 which has compact support and thus its Fourier transform does not contribute to the contour integral $\oint_{\Gamma} dk^0$.

Vectors of the form $\Theta(0, f_1, (\delta \otimes f_2) \circ \rho) \Omega$ will arise below in the following way. Let Λ be a Lorentz boost and $U(\Lambda)$ the corresponding unitary transformation in $\mathcal{H}$. Then

$$(\text{VII.7}) \quad (\theta(0, f)\Omega, E(\lambda N)\theta(0, g)\Omega)$$

$$= (U(\lambda)\theta(0, f)\Omega, E(N)U(\lambda)\theta(0, g)\Omega)$$

$$= (\theta(0, f_1, (\delta \circ f_2) \circ \lambda)\Omega, E(N)\theta(0, g_1, (\delta \circ g_2) \circ \lambda)\Omega)$$

But $(\delta \circ f_2) \circ \lambda = (\delta \circ f'_2) \circ \rho$, $f'_2(x) = cf_2(cx)$ for some $c > 0$ and some rotation ρ .

Next we choose an element μ in the set M of poles of $S(\lambda; \chi)$ or $R(\lambda; \chi)$ on the first sheet of $\mathcal{D}(\delta_1 - \varepsilon)$ together with a neighbourhood $N_\mu = (\mu - \varepsilon', \mu + \varepsilon')$ which does not contain the other pole. By E_j we denote the orthogonal projections onto the subspaces of $E(N_\mu \times \mathbb{R})H$ on which the Poincaré group acts irreducibly. The E_j commute with H, P and each $E(N_{\varepsilon'})$, $N_{\varepsilon'} = N_\mu \times (-\varepsilon', \varepsilon') \subset M$. Let us now take $\Delta = 2m_0$. On linear combinations of vectors of the form $\theta(t, f)\Omega$ or $\theta(0, f_1, (\delta \circ f_2) \circ \rho)\Omega$ with $f_2 \in C_0^\infty(\mathbb{R})$ we can define nonnegative bilinear forms

$$(\varphi, \psi)_{k^1} = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} (\varphi, E(N_\mu \times (k^1 - \varepsilon, k^1 + \varepsilon)) \psi)$$

for $k^1 \in (-\varepsilon', \varepsilon')$, since $(1 \otimes \tilde{f}_2) \circ \rho \in A_j$ for $f_2 \in C_0^\infty(\mathbb{R})$.

The product $(\dots)_{k=0}^1$ has the following properties :

$$\alpha) (E_i \psi, E_j \psi)_0 = 0 \text{ for } i \neq j$$

$\beta)$ For every j there is an $f \in \mathbb{R} \times C_0^\infty(\mathbb{R})$ and a rotation ρ such that

$$(E_j \theta(0, f_1, (\delta \otimes f_2) \circ S) \Omega, E_j \theta(0, f_1, (\delta \otimes f_2) \circ S) \Omega)_0 > 0.$$

For $\beta)$ we use [GJS] that vectors of the form (V II.1) are dense in $E(M)H$ and therefore

$$\|E_j e^{it^0 H} e^{it^1 P} E(N_\epsilon) \theta(0, f) \Omega\| > 0 \text{ for some } f \in C_0^\infty(\mathbb{R})$$

and some $t^0, t^1 \in \mathbb{R}$. Thus $\|E(N_\epsilon) E_j \theta(0, f) \Omega\| > 0$ which by (V II.3) implies that $(E_j \theta(0, f) \Omega, E_j \theta(0, f) \Omega)_{k=1} > 0$ for some $k^1 \in (-\epsilon', \epsilon')$. From this $\beta)$ follows by applying (V II.7) and (V II.6).

As a consequence of $\alpha)$ and $\beta)$ the numbers of particles with mass μ is bounded by the rank of the bilinear form

$$\begin{aligned} \langle f, g \rangle_0 &= \frac{1}{2\pi i} \oint_{\Gamma} d\lambda \left[\langle f_2, R_{22}(\lambda; \chi) g_2 \rangle \right. \\ &\quad \left. + f_1 \langle R_{21}(\lambda; \chi), g_2 \rangle + \langle R_{21}(\lambda; \chi), f_2 \rangle g_1 + f_1 S(\lambda; \chi) g_1 \right] \end{aligned}$$

on $\mathbb{R} \oplus A_0 \times \mathbb{R} \oplus A_0$. Γ is a simple complex curve around μ .

Now we assume that μ is a pole of $S(\lambda; \chi)$.

$\tau = \frac{1}{2\pi i} \oint_{\Gamma} d\chi S(\lambda; \chi) \neq 0$. Then using Proposition 31

$$\begin{aligned} \langle f, g \rangle_0 &= \tau [\langle \varphi, f_2 \rangle \langle \varphi, g_2 \rangle + f_1 \langle \varphi, g_2 \rangle \\ &\quad + \langle \varphi, f_2 \rangle g_1 + f_1 g_1] \\ &= \tau [\langle \varphi, f_2 \rangle + f_1] \cdot [\langle \varphi, g_2 \rangle + g_1] \end{aligned}$$

with $\varphi = R(\lambda, \mu) L(\lambda, \mu, \cdot) \in A_s^*$.

Thus $\langle f, g \rangle_0 \neq 0$ unless $\langle f, f \rangle_0 = 0$ or $\langle g, g \rangle_0 = 0$ i.e.

$\langle \cdot, \cdot \rangle_0$ is rank one.

By Lemma 27 and Lemma 29 the possibilities for M are

$$M = \{\omega(\lambda)\}, \{\omega(\lambda), \chi_1(\lambda)\} \text{ and } \{\omega(\lambda), \chi_1(\lambda), \chi_2(\lambda)\}.$$

The first case is already treated. To complete the discussion of the second one (i.e. $\alpha_n < 0$, $L \equiv 0$) let $\mu = \chi_1(\lambda)$. Then the product $\langle \cdot, \cdot \rangle_0$ reduces to

$$\begin{aligned} \langle f, g \rangle_0 &= \frac{1}{2\pi i} \oint_{\Gamma} d\chi \langle f_2, R(\lambda; \chi) g_2 \rangle = \frac{1}{2\pi i} \oint_{\Gamma} d\chi \langle f_2, S_1(\lambda; \chi) g_2 \rangle \\ &= r(\lambda, \mu) \langle (1 + T_2(\lambda, \mu)^*)^{-1} \varepsilon_0, f_2 \rangle \langle (1 + T_2(\lambda, \mu)^*)^{-1} \varepsilon_0, g_2 \rangle \end{aligned}$$

by Lemma 27, which is also rank one.

Finally consider $M = \{\omega(\lambda), \chi_1(\lambda), \chi_2(\lambda)\}$ and $\mu = \chi_1(\lambda)$.

By Lemma 29 $S(\lambda; \chi)$ is analytic at μ and by (V II.3) this implies that $E(N_e)(1 - E_0)\phi(0)\Omega = 0$ and thus also that $R_{21}(\lambda; \chi)$ is analytic at μ . The product $\langle \dots \rangle_0$ reduces to

$$\langle f, g \rangle_0 = \frac{1}{2\pi i} \oint_{\gamma} d\chi \langle f_z, R_{22}(\lambda; \chi) g_z \rangle$$

and we will show that its rank is zero. To do this it suffices to show that the functions $(\zeta - \zeta_1(\lambda)) \langle f_z, R_{22}(\lambda; \zeta) g_z \rangle$ which are analytic in a neighbourhood of $\zeta = \zeta_1(\lambda)$ are of order $O(\zeta - \zeta_1(\lambda))$. By Lemma 27, Proposition 28 and (VI.4), using Dirac notation we can write

$$\begin{aligned} R' &:= (\zeta - \zeta_1) \hat{R} = |x\rangle \langle x| + O(\zeta - \zeta_1) \\ S' &:= (\zeta - \zeta_1)^{-1} \hat{S} = C(-\langle L|x\rangle^2 C + O(\zeta - \zeta_1))^{-1} \\ &= -\langle L|x\rangle^{-2} + O(\zeta - \zeta_1) \end{aligned}$$

with $|x\rangle = |x\rangle(\lambda; \zeta) = \hat{F}(\lambda; \zeta)^{1/2} (1 + \hat{T}_2(\lambda; \zeta)^*)^{-1} \varepsilon_0$.

Note that $|L\rangle$ and $|X\rangle$ are analytic in the considered region and that $\langle L|X\rangle \neq 0$ at $\zeta = \zeta_1$ by assumption b) and since $\chi_2(\lambda) \in M$. Together with Proposition 31 b) this leads to

$$\begin{aligned} (\zeta - \zeta_1) \langle f_z, \hat{R}_{22} g_z \rangle &= \langle f_z, R' g_z \rangle + \langle R' L, f_z \rangle S' \langle R' L, g_z \rangle \\ &= \langle f_z | x \rangle \langle x | g_z \rangle + O(\zeta - \zeta_1) \\ &\quad - \langle L | x \rangle \langle x | f_z \rangle \langle L, x \rangle^{-2} \langle L | x \rangle \langle x | g_z \rangle + O(\zeta - \zeta_1) \\ &= O(\zeta - \zeta_1) \end{aligned}$$

□

VIII THE BOUND STATE

The discussion in the preceding section shows that under the conditions

$$\begin{aligned} \text{a) } & K(\lambda; \chi, p, q) = O(\lambda^n) \\ & \alpha_n := (n!)^{-1} \partial_\lambda^n K(0; 2\omega_0, 0, 0) \neq 0 \end{aligned}$$

$$\begin{aligned} \text{b) } & L \equiv 0 \\ & L(\lambda; \chi, p) = O(\lambda^m) \\ & \beta_m := (m!)^{-1} \partial_\lambda^m L(0; 2\omega_0, 0) \neq 0 \end{aligned}$$

$$\text{c) } \delta := -\frac{1}{3} (2\bar{u})^{-1} \omega_0^{-2} \beta_m^2 \alpha_n^{-1} \neq 1$$

a) $\lambda P(\phi)_2$ theory with given small $\lambda \geq 0$ can have either no two particle bound state (particle with mass $< 2m(\lambda)$), or one with mass $\chi_1(\lambda)$ (the pole of $R(\lambda; \chi)$; if $L \equiv 0$) or one with mass $\chi_2(\lambda)$ (the pole of $S(\lambda; \chi)$ if $L \neq 0$). In the latter case $\chi_1(\lambda)$ is a C.D.D. zero of $S(\lambda; \chi)$. In order to be sure that there is really a particle with mass $m_B(\lambda) = \chi_1(\lambda)$ or $\chi_2(\lambda)$ we must find a vector $\Psi \in H$ for which

$$(\Psi, E((\mu - \varepsilon, \mu + \varepsilon) \times D^1) \Psi) > 0$$

for some N^1 and arbitrary small $\varepsilon > 0$. The simplest candidates for Ψ are $\hat{\phi}(f)$ and $:\hat{\phi}^2:(f)$ for functions f whose Fourier transform is different from zero somewhere on the mass shell $p^2 = \mu^2$. In this case it suffices to show that $Z_1(\lambda)$

or $z_2(\lambda)$, defined by

$$(VII.1) \quad (\hat{\psi}(x)\Omega, \hat{\psi}(y)\Omega) = z(\lambda) \Delta_+(\langle x-y \rangle, w^2(\lambda)) \\ + z_2(\lambda) \Delta_+(\langle x-y \rangle, w_B^2(\lambda)) + \int_{4w^2(\lambda)}^{\infty} d\mu_1(a) \Delta_+(\langle x-y \rangle, a)$$

$$(VII.2) \quad (:\hat{\psi}^2:(x)\Omega, :\hat{\psi}^2:(y)\Omega) = z_3(\lambda) \Delta_+(\langle x-y \rangle, w^2(\lambda)) \\ + z_1(\lambda) \Delta_+(\langle x-y \rangle, w_B^2(\lambda)) + \int_{4w^2(\lambda)}^{\infty} dS_\lambda(a) \Delta_+(\langle x-y \rangle, a)$$

are different from zero. Equation (VII.2) follows from the fact that $:\phi^2:(x)$ exists as relatively local fields [Sch].

Theorem 33 :

Under the conditions a)b)c) and for $\lambda \in (0, \lambda_0)$ with λ_0 sufficiently small we have

- 1) If $\alpha_n > 0$ or $\alpha_n < 0$ with $n = 2m$ and $\gamma > 1$ then there is no two particle bound state.
- 2) In the remaining cases there is exactly one two particle bound state.

Its mass $m_B(\lambda)$ is C^∞ in λ with

$$w_B^2(\lambda) = 4w^2(\lambda) - (1 - \lambda^{2u-n}\delta)^2 w_c^2 \left(\frac{1}{u} w_c^{2(u-1)} \alpha_n \right)^2 \left(\frac{\lambda}{w_c} \right)^{2u} \\ + O\left(\left(\frac{\lambda}{w_c}\right)^{2u+1}\right).$$

For $L \equiv 0$ $z_1(\lambda)$ is C^∞ in λ and

$$z_1(\lambda) = -(\bar{u} \omega_c^{2(u-1)} \alpha_n) \left(\frac{\lambda}{\omega_c^2}\right)^n + O\left(\left(\frac{\lambda}{\omega_c^2}\right)^{n+1}\right).$$

If $L \neq 0$ $z_2(\lambda)$ is C^∞ in λ and

$$z_2(\lambda) = -\frac{1}{9}(1-\lambda^{2u+n}\delta)(\bar{u} \omega_c^{2(u-1)} \alpha_n)(\omega_c^{2(u-1)} \beta_n)^2 \left(\frac{\lambda}{\omega_c^2}\right)^{2u+n} + O\left(\left(\frac{\lambda}{\omega_c^2}\right)^{2u+n+1}\right).$$

For example if $c_4 < 0$ then

$$(u = 1, \alpha_1 = 6(2\pi)^{-1}c_4, \text{ see (V.7)})$$

$$\omega_B^2(\lambda) = 4\omega^2(\lambda) - 9c_4^2 \omega_c^2 \left(\frac{\lambda}{\omega_c^2}\right)^2 + O\left(\left(\frac{\lambda}{\omega_c^2}\right)^3\right).$$

The effect of a $c_3\phi^3$ - term is an increase of $m_B^2(\lambda)$ by

$$-9c_3c_4^2 \omega_c^2 \left(\frac{\lambda}{\omega_c^2}\right)^3 + O\left(\left(\frac{\lambda}{\omega_c^2}\right)^4\right)$$

($m = 1, \beta_1 = 3c_3$, see (V.7)). This cannot remove a bound state unless $c_4 = c_4(\lambda) = 0(\lambda)$.

Proof of Theorem 33 :

For $L \equiv 0$ we can follow the analysis of [DE]. In this case $m_B(\lambda) = \chi_1(\lambda)$, the pole of $R(\lambda; \chi)$, which is the zero of the functions $H_1(\lambda; \chi)$

$$\hat{H}_1(\lambda; \chi) = \zeta + \hat{Q}(\lambda; \chi) = \zeta + \hat{\Gamma}_0(\lambda; \chi) \langle \varepsilon_0, (1 + \hat{T}_2(\lambda; \chi))^{-1} \hat{K}(\lambda; \chi) \varepsilon_0 \rangle$$

(see the proof of Lemma 27). By Lemma 24 and Lemma 26 $\hat{H}_1(\lambda; \zeta)$ is C^∞ in λ and holomorphic in $\hat{\mathcal{D}}(\delta_1 - \varepsilon)$. Moreover $H_1(0,0) = 0$ and $\partial_\zeta \hat{H}_1(0,0) = 1$. Thus by an implicit function theorem there is a C^∞ function $\zeta_1(\lambda)$ defined for small $\lambda \geq 0$ satisfying $\zeta_1(0) = 0$ and $\hat{H}_1(\lambda; \zeta_1(\lambda)) = 0$.

Thus $\chi_1^2(\lambda) = 4m^2(\lambda) - \zeta_1^2(\lambda)$ is C^∞ in small $\lambda \geq 0$. Since $\hat{H}_1(\lambda; \zeta) = O(\lambda^n)$ the first derivative $\partial_\lambda^k \chi_1^2(0)$ different from $4\partial_\lambda^k m^2(0)$ is

$$\begin{aligned} \partial_\lambda^{2n} \chi_1^2(0) &= 4\partial_\lambda^{2n} m^2(0) - \frac{(2n)!}{(n!)^2} (\partial_\lambda^n \zeta_1(0))^2 \\ &= 4\partial_\lambda^{2n} m^2(0) - (2n)! m_0^{-2} (\bar{u} m_0^{-2} \alpha_n)^2 \end{aligned}$$

where we have used that

$$\begin{aligned} (\text{VII.3}) \quad \partial_\lambda^n \zeta_1(0) &= - \frac{\partial_\lambda^n \hat{H}_1(0,0)}{\partial_\zeta \hat{H}_1(0,0)} = - \partial_\lambda^n \hat{H}_1(0,0) \\ &= -\hat{\Gamma}_0(0;0) n! \alpha_n = -\bar{u} m_0^{-1} n! \alpha_n \end{aligned}$$

So we obtain

$$(\text{VII.4}) \quad \chi_1^2(\lambda) = 4m^2(\lambda) - m_0^2 (\bar{u} m_0^{-2} \alpha_n)^2 \left(\frac{\lambda}{m_0^2}\right)^{2n} + O\left(\left(\frac{\lambda}{m_0^2}\right)^{2n+1}\right)$$

In the notation of Section V II.

$:\hat{\phi}^2:(0) = \Theta(0, (0, \delta))$ which leads to

$$(\hat{\psi}^2(0)\Omega, e^{-ix^0/H} e^{ix^4/P} \hat{\psi}^2(0)\Omega) = \langle \zeta \zeta^1_{zz}(0,0; x, x) \rangle$$

and together with (VIII.2)

$$\frac{(\zeta\bar{u})^{-1}\zeta_3(\lambda)}{u^2(\lambda)-\chi^2} + \frac{(\zeta\bar{u})^{-1}\zeta_4(\lambda)}{\chi_1^2(\lambda)-\chi^2} + \int \frac{(\zeta\bar{u})^{-1}d\mu_\lambda(q)}{q^2-\chi^2} = (\zeta\bar{u})^{-2} \langle 1, R_{22}(\lambda; \chi) 1 \rangle.$$

Consequently in the case $L \equiv 0$ this allows to compute $\zeta_1(\lambda)$ as follows

$$\begin{aligned} \zeta_1(\lambda) &= (\zeta\bar{u})^{-1} \lim_{\chi \rightarrow \chi_1(\lambda)} (\chi_1^2(\lambda) - \chi^2) \langle 1, R(\lambda; \chi) 1 \rangle \\ &= (\zeta\bar{u})^{-1} \lim_{\zeta \rightarrow \zeta_1(\lambda)} (\zeta^2 - \zeta_1(\lambda)^2) \langle 1, \hat{R}(\lambda; \zeta) 1 \rangle \\ &= (\zeta\bar{u})^{-1} 2 \zeta_1(\lambda) \lim_{\zeta \rightarrow \zeta_1(\lambda)} (\zeta - \zeta_1(\lambda)) \langle 1, \hat{S}_1(\lambda; \zeta) 1 \rangle \\ &= \pi^{-1} \zeta_1(\lambda) \lim_{\zeta \rightarrow \zeta_1(\lambda)} \hat{T}_0(\lambda; \zeta) \langle \varepsilon_0, (1 + \hat{T}_2(\lambda; \zeta))^{-1} 1 \rangle^2 \left[\frac{\hat{H}_2(\lambda; \zeta)}{\zeta - \zeta_1(\lambda)} \right] \\ &= \pi^{-1} \zeta_1(\lambda) \hat{T}_0(\lambda; \zeta_1(\lambda)) \langle \varepsilon_0, (1 + \hat{T}_2(\lambda; \zeta_1(\lambda)))^{-1} 1 \rangle^2 \\ &\quad \times \left[\partial_\zeta \hat{H}_2(\lambda; \zeta_1(\lambda)) \right]^{-1} \end{aligned}$$

where we have used Lemma 27. By (VIII.3) and the fact that

$$\partial_{\zeta} \hat{H}_1(\lambda; \zeta_1(\lambda)) = 1 + O(\lambda) \quad \text{it follows that} \quad \partial_{\lambda}^n z_1(0) = -\Pi m_0^{-2} n! \alpha_n.$$

$$\text{In the case } L \neq 0 \quad m_B(\lambda) = \chi_2(\lambda) = \sqrt{4m^2(\lambda) - \zeta_2(\lambda)^2}$$

where $\zeta_2(\lambda)$ is the pole of $\hat{S}(\lambda; \zeta)$ i.e. the solution of (VI.14) i.e. the zero of

$$\begin{aligned} (\text{VII.5}) \quad \hat{H}_2(\lambda; \zeta) &= \zeta_1(\lambda) (1 - \lambda^{2m-u} \delta(\lambda)) - \zeta \\ &+ [\lambda^n (\hat{q}(\lambda; \zeta) - \hat{q}(\lambda; \zeta_1(\lambda))) - \lambda^{2m} (\delta(\lambda; \zeta) - \delta(\lambda; \zeta_1(\lambda))) Q(\lambda)] . \end{aligned}$$

$\hat{H}_2(\lambda; \zeta)$ is C^∞ in λ and analytic in ζ in some neighbourhood $[0, \lambda'] \times N$ of $(0, 0)$. This follows from analogous statements for $\hat{q}(\lambda; \zeta)$ and $\gamma(\lambda; \zeta)$ made in the proof of Lemma 29. Since $\hat{H}_2(0, 0) = 0$ and $\partial_{\zeta} \hat{H}_2(0, 0) = -1$ we can conclude that the function $\zeta_2(\lambda)$ which solves $\hat{H}_2(\lambda; \zeta_2(\lambda)) = 0$ for small $\lambda > 0$ is C^∞ . Furthermore from (VIII.5) it follows immediately that

$$\begin{aligned} \zeta_1(\lambda) - \zeta_2(\lambda) &= \lambda^{2m-u} \delta(\lambda) \zeta_1(\lambda) [1 + O(\lambda^u)] \\ &= \lambda^{2m-u} \delta \zeta_1(\lambda) + O(\lambda^{2m+1}) \end{aligned}$$

which leads to

$$\begin{aligned} (\text{VII.6}) \quad \chi_2^2(\lambda) &= 4m^2(\lambda) - (1 - \lambda^{2m-u} \delta)^2 (4m^2(\lambda) - \chi_1^2(\lambda)) \\ &+ O(\lambda^{2m+u+1}) , \end{aligned}$$

Combined with (VIII.4) this gives the desired expansion for $m_B^2(\lambda)$.

There remains to consider $Z_2(\lambda)$. By (VIII.1)

$$\begin{aligned} Z_2(\lambda) &= 2\bar{u} \lim_{\chi \rightarrow \chi_2(\lambda)} (\chi^2(\lambda) - \chi^2) S(\lambda; \chi) \\ &= 2\bar{u} \lim_{\zeta \rightarrow \zeta_2(\lambda)} (\zeta^2 - \zeta_2(\lambda)^2) \hat{S}(\lambda; \zeta) \\ &= 4\bar{u} \zeta_2(\lambda) \lim_{\zeta \rightarrow \zeta_2(\lambda)} (\zeta - \zeta_2(\lambda)) \hat{C}(\lambda; \zeta) (1 + \hat{D}(\lambda; \zeta)) (1 - \hat{F}_2(\lambda; \zeta))^{-1} \\ &= -4\bar{u} \zeta_2(\lambda) C(\chi_2(\lambda)) (1 + D(\lambda; \chi_2(\lambda))) [\partial_\zeta \hat{F}_2(\lambda; \zeta_2(\lambda))]^{-1}. \end{aligned}$$

The numerator and denominator in the last expression are O^∞ for small $\lambda \geq 0$ (see the proof of Lemma 29). By differentiating

$$\hat{F}_2(\lambda; \zeta) = \lambda^{2u} (2\bar{u})^2 (1 + \hat{D}(\lambda; \zeta)) \hat{C}(\lambda; \zeta) \hat{t}(\lambda; \zeta) (\zeta - \lambda^u \hat{q}(\lambda; \zeta))^{-1}$$

and using that $\hat{F}_2(\lambda; \zeta_2(\lambda)) = 1$ we obtain

$$[\partial_\zeta \hat{F}_2(\lambda; \zeta_2(\lambda))]^{-1} = -\lambda^{2u} (2\bar{u})^2 C(\chi_2(\lambda)) t(\lambda; \chi_2(\lambda)) + O(\lambda^{2u+1})$$

such that $Z_2(\lambda)$ turns out to be O^∞ in small $\lambda > 0$. It suffices to take the definition of $\hat{t}(\lambda; \zeta)$ and the result for $\zeta_2(\lambda)$ to obtain the expansion

$$\begin{aligned} (\text{VIII.7}) \quad Z_2(\lambda) &= \lambda^{2u} (2\bar{u})^2 2 \zeta_2(\lambda) C(2u_0)^2 t(0, 2u_0) + O(\lambda^{2u+u+1}) \\ &= -\frac{1}{9} (1 - \lambda^{2u-u}) (\bar{u} u_0^{2(u-1)} \alpha_u) (u_0^{2(u-1)} \beta_u)^2 \left(\frac{\lambda}{u_0^2}\right)^{2u+u} \\ &\quad + O\left(\left(\frac{\lambda}{u_0^2}\right)^{2u+u+1}\right). \end{aligned}$$

□

IX A $P(\phi)_2$ -MODEL WITH STRONG EXTERNAL FIELD

In this section we will apply the methods of the parts I-VIII to the $P(\phi)_2$ -model with interaction density

$$\bar{P}(\varphi) = \lambda \varphi^4 - 4\mu \varphi, \quad \lambda > 0, \quad \mu^2 \text{ large.}$$

In [SI] Spencer showed that the objects

$$\begin{aligned} \bar{S}_{\phi^{n_1}, \dots, \phi^{n_n}}(\lambda; x_1, \dots, x_n) &= \lim_{h \rightarrow 1} \left\langle \frac{n}{h} : \phi^{n_i}(x_i) \right\rangle_{h, \bar{P}, \varepsilon^2} \\ &= \lim_{h \rightarrow 1} \frac{\int d\mu_{\varepsilon^2} \frac{n}{h} : \phi^{n_i}(x_i) e^{-\int d^2x h(x) : \bar{P}(\phi) : (x)}}{\int d\mu_{\varepsilon^2} e^{-\int d^2x h(x) : \bar{P}(\phi) : (x)}} \end{aligned}$$

define Schwinger functions from which a Wightman field theory can be constructed. We will follow here the proof of [SI]. It is based on a transformation $\bar{P} \rightarrow P$ such that

$$\bar{S}(X) = \sum_{Y \prec X} \varepsilon^{|Y| - |X|} S(\varepsilon^{-1} Y)$$

where $\varepsilon^{-1} = (\frac{\lambda}{\mu})^{1/3}$ is the value of x which minimizes $\bar{P}(x)$, and $S(\dots)$ are the Schwinger functions obtained with an interaction polynomial $P(\phi)$ which represents a weak coupling in the sense that

(IX.1) The coefficients of P are small compared with the transformed bare mass 12λ and

$$(IX.2) \quad \int d\mu_{m_1} e^{-\int d^2x: P(\phi): (x)} \leq e^{k|\Lambda|}, \quad k > 0,$$

with $k \rightarrow 0$ as $|\mu| \rightarrow \infty$.

For large $\lambda > 0$ (independent of μ) these conditions are sufficient for the convergence of the cluster expansions [GJS], [S II]. For general $\lambda > 0$ the same results are obtained after an additional scaling. The mentioned transformation is composed of three parts :

a) Translation $\phi(x) \rightarrow \phi(x) + \varepsilon^{-1}$

leads to $\langle A \rangle_{h, \bar{P}, \varepsilon^2} = \langle A^{\varepsilon^{-1}} \rangle_{h, \bar{P}, \varepsilon^2}$

where $A^a(\phi) = A(\phi + a)$

and
$$P_0(x) = \bar{P}(x + \varepsilon^{-1}) - \bar{P}(\varepsilon^{-1}) + \varepsilon^{-1} \bar{G}^2 x \\ = \lambda x^4 + 4\lambda \varepsilon^{-1} x^3 + 6\lambda \varepsilon^{-2} x^2 + \varepsilon^{-1} \bar{G}^2 x$$

with leading mass term $6\lambda \varepsilon^{-2} x^2$.

This motivates

b) Scaling $x \rightarrow \varepsilon^{-1} x$.

We have $\langle A^{\varepsilon^{-1}} \rangle_{h, \bar{P}_0, \varepsilon^2} = \langle A^{\varepsilon^{-1}} \rangle_{h(\varepsilon^{-1}), \varepsilon^2 \bar{P}_0, \varepsilon^2 \varepsilon^2}$

where $A^a_{a'}(\phi) = A^a(\phi(a'))$,

$$\begin{aligned} \varepsilon^2 \mathcal{P}_0(x) &= \lambda \varepsilon^2 x^4 + 4\lambda \varepsilon x^3 + \left(6\lambda - \frac{1}{2} \varepsilon^2 \varepsilon^2\right) x^2 \\ &\quad + \frac{1}{2} \varepsilon^2 \varepsilon^2 x^2 - \varepsilon \varepsilon^2 x \\ &\stackrel{\text{def.}}{=} Q_\ell(x) + \left(\ell\lambda - \frac{1}{2} \varepsilon^2 \varepsilon^2\right) x^2, \quad 1 \leq \ell \leq 6. \end{aligned}$$

Quadratic terms can be taken into the bare mass by using

c) Mass shift

$$\langle A^{\varepsilon^{-1}} \rangle_{h(\varepsilon), Q_\ell + (\ell\lambda - \frac{1}{2} \varepsilon^2 \varepsilon^2), \varepsilon^2 \varepsilon^2} = \langle A^{\varepsilon^{-1}} \rangle_{h(\varepsilon), \varphi_\ell, 2\ell\lambda}$$

where

$$:\mathcal{P}_\ell:_{2\ell\lambda} = :Q_\ell:_{\varepsilon^2 \varepsilon^2} + C(\ell),$$

and $\langle \dots \rangle_{h, \dots, m^2, d\mu_{m^2}, :A:_{m^2}}$ are defined with respect to the covariance $C_{m^2} = (-\Delta + m^2 h + 2\lambda(1-h))^{-1}$. Wick ordering with respect to two different covariances is related by

$$\begin{aligned} :\phi^m:_{m_0^2}(x) &= \sum_{k \geq 0}^{[n/2]} C_{n,k}^e :\phi^{n-2k}:_{m_0^2}(x) \\ C_{n,k}^e &= \frac{n!}{2^k k! (n-2k)!} [C_{m_0^2} - C_{m_\ell^2}](x, x)^k. \end{aligned}$$

with $m_0^2 = b^2 \varepsilon^2$, $m_\ell^2 = 2\ell\lambda$ and $C(\ell) = (6-\ell) (\Phi^2:_{m_\ell^2} - \Phi^2:_{m_0^2})$

we get

$$\begin{aligned} (\text{IX.3}) \quad \mathcal{P}_\ell(x) &= \varepsilon^2 \lambda x^4 + 4\lambda \varepsilon x^3 + (6-\ell)\lambda x^2 + \frac{1}{2} \varepsilon^2 \varepsilon^2 x^2 - \varepsilon \varepsilon^2 x \\ &\quad + \varepsilon^2 \lambda C_{4,2}^e x^2 + 4\lambda \varepsilon C_{3,1}^e x + \varepsilon^2 \lambda C_{4,2}^e + \frac{1}{2} \varepsilon^2 \varepsilon^2 C_{2,1}^e. \end{aligned}$$

Since $C_{nk}^e = O((\log|\varepsilon|)^2)$ the coefficients of $P = P_6$ are small compared with $m_0^2 = 12\lambda$. In order to prove the bound (IX.2) we show first

$$(IX.4) \quad P_2(x) \geq \frac{1}{10} \lambda \varepsilon^2 (1 - |\delta_1 \varepsilon|^{1/4}) x^4 - 2\delta_1 - \delta_0$$

with $\delta_1(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then by standard estimates [DG]

$$\int d\mu_{m_i} e^{-\int \lambda d^2x : P_2(\phi) : (x)} \leq e^{N|\lambda|}$$

for some $N < \infty$ uniformly in $|\varepsilon| \leq \varepsilon_0$, and thus

$$\begin{aligned} & \int d\mu_{m_i} e^{-\int \lambda d^2x : P_2(\phi) :_{m_i}} \\ &= \frac{\int d\mu_{m_i} e^{-\int \lambda d^2x [: P_2(\phi) :_{m_i}(x) + S : \phi^2 :_{m_i}(x)]}}{\int d\mu_{m_i} e^{-\int \lambda d^2x S : \phi^2 :_{m_i}(x)}} \\ &\leq \int d\mu_{m_i} e^{-\int \lambda d^2x : P_2(\phi) :_{m_i}^2(x)} \quad \text{by Jensen's inequality} \\ &\leq e^{N|\lambda|} \end{aligned}$$

In fact N can be replaced by $k(\varepsilon) = o(1)$ since the coefficients of P_6 vanish as $\varepsilon \rightarrow 0$. Finally P_1 can be bounded as follows ($\varepsilon > 0$) :

$$\begin{aligned}
 (\text{IX.5}) \quad \varepsilon^2 \bar{P}_1\left(\frac{x}{\varepsilon}\right) &= \lambda x^4 + 4\lambda x^3 + (5\lambda - \delta_2)x^2 - \delta_1 \varepsilon^{1/5} x - \delta_0 \varepsilon^2 \\
 &= \frac{1}{10} (1 - \delta_1^{5/4} \varepsilon^{3/5}) \lambda x^4 + \left[\frac{9}{10} \lambda x^4 + 4\lambda x^3 + (5\lambda - \delta_2)x^2 \right] \\
 &\quad + \frac{1}{10} \delta_1^{5/4} \varepsilon^2 \left[(\delta_1^{-1/4} \varepsilon^{-3/5} x)^4 - 10(\delta_1^{-1/4} \varepsilon^{-3/5} x) + 20 \right] \\
 &\quad - 2\delta_1^{5/4} \varepsilon^2 - \delta_0 \varepsilon^2
 \end{aligned}$$

where $\delta_i \rightarrow 0$ as $\varepsilon \rightarrow 0$. This proves (IX.4) since the quantities in square brackets are nonnegative.

In the infinite volume limit the coefficients c_{nk}^e can be calculated explicitly

$$c_{n,k}^e = \frac{n!}{2^k k! (n-2k)!} \left(\frac{1}{4r} \log \frac{w_e^2}{w_0^2} \right)^k.$$

In particular

$$\begin{aligned}
 c_{21}^6 &= -\frac{1}{2} (2\bar{u})^{-1} \log \left(\frac{\bar{v}^2}{12\lambda} \varepsilon^2 \right) \\
 c_{31}^6 &= -\frac{3}{2} (2\bar{u})^{-1} \log \left(\frac{\bar{v}^2}{12\lambda} \varepsilon^2 \right) \\
 c_{41}^6 &= -3 (2\bar{u})^{-1} \log \left(\frac{\bar{v}^2}{12\lambda} \varepsilon^2 \right) \\
 c_{42}^6 &= -\frac{3}{4} (2\bar{u})^{-2} \log^2 \left(\frac{\bar{v}^2}{12\lambda} \varepsilon^2 \right).
 \end{aligned}$$

By using that $\langle A \rangle_{\bar{p}, b^2}^{\varepsilon-1} = \langle A \rangle_{\varepsilon-1}^{\varepsilon-1} \rangle_{\rho, 12\lambda}$ for the infinite volume

expectation we obtain the connection between the corresponding Schwinger functions $\bar{S}(x_1, \dots, x_n)$ and $S(x_1, \dots, x_n)$.

$$\begin{aligned} \langle \bar{u}_{i=1}^n \phi(f_i) \rangle_{\bar{p}, \varepsilon^2} &= \langle \bar{u}_{i=1}^n (\phi \circ \varepsilon^{-1})(f_i) \rangle_{\bar{p}, \varepsilon^2} = \langle \bar{u}_{i=1}^n (\phi \circ \varepsilon^{-1})(\varepsilon^2 f_i(\varepsilon \cdot)) \rangle_{\bar{p}, \varepsilon^2 \lambda} \\ &= \sum_{\substack{I \cup J = \{1, \dots, n\} \\ I \cap J = \emptyset}} \langle \bar{u}_{i \in I} \phi(\varepsilon^2 f_i(\varepsilon \cdot)) \rangle_{\bar{p}, \varepsilon^2 \lambda} \bar{u}_{j \in J} \varepsilon^{-1}(f_j) \end{aligned}$$

with $\varepsilon^{-1}(f) = \varepsilon^{-1} \int d^2 x f(x)$, i.e.

$$\bar{S}(X) = \sum_{Y \subset X} \varepsilon^{|Y|-|X|} S(\varepsilon^{-1} Y)$$

and $\bar{S}(x_1; \dots; x_n) = S(\varepsilon^{-1} x_1; \dots; \varepsilon^{-1} x_n)$ for the truncated functions. Thus with the notation of (IV.1)

$$\bar{S}(k) = \varepsilon^2 S(\varepsilon k), \quad \bar{R}_{22}(k, p, q) = \varepsilon^6 R_{22}(\varepsilon k, \varepsilon p, \varepsilon q)$$

which gives the relation between the masses and the field strengths as defined in (VIII.1), (VIII.2) (if this makes sense).

$$\begin{aligned} \text{(IX.6)} \quad \bar{u}(\varepsilon) &= \varepsilon^{-1} u(\varepsilon), & \bar{z}(\varepsilon) &= z(\varepsilon) \\ \bar{u}_B(\varepsilon) &= \varepsilon^{-1} u_B(\varepsilon), & \bar{z}_1(\varepsilon) &= z_1(\varepsilon) \\ & & \bar{z}_2(\varepsilon) &= z_2(\varepsilon) \end{aligned}$$

So there remains to repeat the analysis of the Sections I-VIII with the coupling constant λ replaced by ε and the polynomial

$\sum_{n=1}^{2N} C_n x^n$ replaced by the polynomial $P(x) = P_6(x)$ given by (IX.3).

Theorem 34 :

In the $\lambda \phi^4 + 4_\mu \phi$ theory with $\lambda > 0$ and sufficiently large $|\mu|$ there is exactly one two particle bound state, with mass satisfying

$$(IX.7) \quad \bar{u}_B^2(\epsilon) = 4\bar{u}^2(\epsilon) - \frac{4g}{12} \lambda \epsilon^2 + O(-\epsilon^4 \log |\epsilon|)$$

$\epsilon = (\frac{\lambda}{\mu})^{1/3}$, in addition to the particle with mass

$$(IX.8) \quad \bar{u}^2(\epsilon) = 12 \lambda \epsilon^{-2} - \frac{6}{4} \lambda \log |\epsilon| + O(1) .$$

There is no other spectrum up to $2m(\epsilon)$ and the residua of the two point function at $\bar{m}^2(\epsilon)$, $\bar{m}_B^2(\epsilon)$ respectively are of the order

$$(IX.9) \quad \bar{Z}(\epsilon) = 1 - (\pi^{-1} - \frac{\sqrt{3}}{3}) \epsilon^2 + O(-\epsilon^4 \log |\epsilon|)$$

$$(IX.10) \quad \bar{Z}_2(\epsilon) = \frac{7}{108} \epsilon^4 + O(-\epsilon^6 \log |\epsilon|) .$$

Furthermore $\bar{m}^2(\epsilon)$, $\bar{m}_B^2(\epsilon)$, $Z(\epsilon)$ and $Z_2(\epsilon)$ are C^∞ in

$\epsilon \in (-\epsilon_0, 0) \cup (0, \epsilon_0)$ for some $\epsilon_0 > 0$ and their expansions

(IX.7) - (IX.10) are asymptotic.

Proof :

Notice that some coefficients of our polynomial P have logarithmic singularities at $\varepsilon = 0$. Therefore we should expand the quantities $m(\varepsilon), Z(\varepsilon), \dots$ in powers of ε and $\log|\varepsilon|$ rather than in powers of ε only. To do this with the methods of the previous sections it is profitable to introduce a two parameter family $P_{(z)}$ of polynomials such that $P_{(-\log|\varepsilon|)} = P$, namely

$$\begin{aligned} P_{(z)}(x) = & \lambda \varepsilon^2 x^4 + 4\lambda \varepsilon x^3 \\ & + \varepsilon^2 \left(3(z\bar{u})^{-1} \lambda \log\left(\frac{\bar{u}^2}{12\lambda}\right) + \frac{1}{2} \bar{u}^2 \right) x^2 \\ & + 2\varepsilon \left(3(z\bar{u})^{-1} \lambda \log\left(\frac{\bar{u}^2}{12\lambda}\right) + \frac{1}{2} \bar{u}^2 \right) x \\ & + 6(z\bar{u})^{-1} \varepsilon^2 \lambda \bar{u}^2 x^2 \\ & + 12(z\bar{u})^{-1} \varepsilon \lambda \bar{u}^2 x \end{aligned}$$

In the region $|z| < -2\log|\varepsilon|$, ε^2 small, (IX.1) holds also for $P_{(z)}$. Furthermore (IX.4) remains true when P_1 is replaced by $\text{Re}P_{1,(z)}$, where $:P_{1,(z)}(\Phi) = S\Phi^2:_{2\lambda} = :P_{(z)}(\Phi):_{12\lambda}$. Consequently the cluster expansions [GJS], [S II] for interactions $P_{(z)}$ converge and define Schwinger functions which are C^∞ in ε and analytic in z for $|z| < -2\log|\varepsilon|$, ε^2 small. It is now straightforward to repeat the Sections I-VIII for

$$P_{(2)}(x) = \varepsilon \sum_{n=1}^4 c_n(\varepsilon, z) x^n \quad \text{and real } z.$$

We obtain (see Proposition 25 and (V.4))

$$\begin{aligned} (\text{IX.11}) \quad u_{(2)}^2(\varepsilon) &= 12\lambda + 2c_2\varepsilon - [6c_3c_1(z\bar{u})k_1(0) \\ &\quad + 9c_3^2 \cdot 2(z\bar{u})k_2(\langle\sqrt{-12\lambda}, 0\rangle)]\varepsilon^2 + O(\varepsilon^4) \\ &= 12\lambda + \frac{6}{\pi}\lambda z\varepsilon^2 + \text{const.} \cdot \varepsilon^2 + O(\varepsilon^4) \end{aligned}$$

$$(\text{IX.12}) \quad z_{(2)}(\varepsilon) = 1 - 9c_3^2 3!(z\bar{u})\partial_{x^2}k_2(\langle\sqrt{-12\lambda}, 0\rangle)\varepsilon^2 + O(\varepsilon^4).$$

The coefficient of ε^2 is equal to

$$\begin{aligned} &-108(z\bar{u})^{-1}c_3^2\partial_{x^2}[(48\lambda-x^2)^{-\frac{1}{2}}\arcsin(\frac{x^2}{48\lambda})^{\frac{1}{2}}\cdot x^{-1}]_{x^2=12\lambda} \\ &= -108(z\bar{u})^{-1}(4\lambda)^2(48\lambda)^{-\frac{1}{2}}\partial_x[(1-x^2)^{-\frac{1}{2}}\frac{\arcsin x}{x}]_{x=\frac{1}{2}} \\ &= -(\frac{-1}{\pi} - \frac{\sqrt{3}}{9}) . \end{aligned}$$

By using (V.7) we can compute the first terms in the expansions of $L_{(2)}$ and $K_{(2)}$

$$L_{(2)}(\varepsilon; k, p) = -12\lambda\varepsilon + O(\varepsilon^2)$$

$$\begin{aligned} K_{(2)}(\varepsilon; k, p, q) &= 6(z\bar{u})^{-1}c_4\varepsilon - 18(z\bar{u})^{-1}(\langle p-q \rangle^2 + 12\lambda)^{-1}c_3^2\varepsilon^2 \\ &\quad + O(\varepsilon^4) \end{aligned}$$

so that, in the terminology of Section VIII we are in the case where $n = 2m$ with

$$\begin{aligned}\beta_1 &= -12\lambda \\ \alpha_2 &= -18(z\bar{u})^{-1}\lambda < 0 \\ \gamma &= -\frac{1}{3}(z\bar{u})^{-1}(12\lambda)^{-1}\beta_1^2\alpha_2^{-1} = \frac{2}{9}\end{aligned}$$

which leads by Theorem 33 to a bound state with mass

$$(IX.13) \quad u_{B,(z)}^z(\varepsilon) = 4u_{(z)}^z(\varepsilon) - \frac{49}{12}\lambda\varepsilon^2 + O(\varepsilon^4)$$

and the corresponding field strength is given by

$$(IX.14) \quad z_{z,(z)}(\varepsilon) = \frac{7}{108}\varepsilon^4 + O(\varepsilon^6).$$

This is true for all $P_{(z)}$ -theories separately.

But we can repeat the same program (except statements about the reality of zeros or any connection to a physical Hilbert space) with uniform bounds for small ε^2 and for z in some complex neighbourhood of $[\log|\varepsilon|, -\log|\varepsilon|]$. In this way the $O(\varepsilon^n)$ -rests in the Taylor expansions above can be bounded uniformly by $\text{const. } \varepsilon^{n+2}|\log|\varepsilon||$. Furthermore the quantities $m_{(z)}^2(\varepsilon)$, $z_{(z)}(\varepsilon)$, $m_{B,(z)}^2(\varepsilon)$ and $z_{2,(z)}(\varepsilon)$ are not only C^∞ in ε but also analytic in z . This implies that $m^2(\varepsilon)$, $z(\varepsilon)$, $m_B^2(\varepsilon)$ and $z_2(\varepsilon)$ are C^∞ in ε for small $\varepsilon^2 > 0$.

□

ACKNOWLEDGEMENTS

It is a pleasure to thank Prof. J.-P. Eckmann for the suggestion of the problems and for his help during the elaboration of this thesis.

I also want to thank Prof. R. Sénéor for his participation in the jury, and Prof. J. Glimm for pointing out an error in an earlier version of this work.

REFERENCES

- [B] J. Bros, Some analytic properties implied by the two-particle structure of Green's functions in general quantum field theory. In : R.P. Gilbert and R.G. Newton (Eds) Analytic methods in mathematical physics. Gordon and Breach, New York - London - Paris, 1970.

- [BL] J. Bros and M. Lassalle, Analytic properties and many-particle structure in general quantum field theory. Commun. Math. Phys. 36, 185 (1974), 43, 279 (1975) 54, 33 (1977).

- [CD] M. Combes and F. Dunlop, n-particle irreducible functions in Euclidean quantum field theory. Preprint.

- [D] J. Dimock, Asymptotic perturbation expansions in the $P(\phi)_2$ quantum field theory. Commun. Math. Phys. 35, 347 (1974).

- [DE I] J. Dimock and J.-P. Eckmann, On the bound state in weakly coupled $\lambda(\phi^6 - \phi^4)_2$. Commun. Math. Phys. 51, 41 (1976).

- [DE II] J. Dimock and J.-P. Eckmann, Spectral properties and bound state scattering for weakly coupled $\lambda P(\phi)_2$ models. Ann. of Phys. Vol. 103, No 2, 289 (1977).
- [DG] J. Dimock and J. Glimm, Measures on Schwartz distribution space and applications to $P(\phi)_2$ field theories. Adv. Math. 12, 58 (1974).
- [EEF] J.-P. Eckmann, H. Epstein and J. Fröhlich, Asymptotic perturbation expansion for the S-matrix and the definition of time ordered functions in relativistic quantum field models. Ann. Inst. Henri Poincaré Vol. XXV, No 1, 1 (1976).
- [EMS] J.-P. Eckmann, J. Magnen and R. Sénéor, Decay properties and Borel summability for the Schwinger functions in $P(\phi)_2$ theories. Commun. Math. Phys. 39, 251 (1975).
- [GJ I] J. Glimm and A. Jaffe, Particles and bound states and progress toward unitarity and scaling. In : H. Araki (Ed) International symposium on mathematical problems in theoretical physics, Springer lecture notes in physics, Vol. 29, Berlin-Heidelberg-New York, (1975).

- [GJS I] J. Glimm, A. Jaffe and T. Spencer, The Wightman axioms and particle structure in the $P(\phi)_2$ quantum field model. Ann. Math. 100, 585 (1974).
- [GJS II] J. Glimm, A. Jaffe and T. Spencer, The particle structure of the weakly coupled $P(\phi)_2$ model and other applications of high temperature expansions. In : G. Velo and A. Wightman (Eds) Constructive quantum field theory, Springer lecture notes in physics, Vol. 25, Berlin - Heidelberg - New York, (1973).
- [OS] K. Osterwalder and R. Sénéor, The scattering matrix is non-trivial for weakly coupled $P(\phi)_2$ models. Helv. Phys. Acta 49, 525 (1976).
- [OSCH] K. Osterwalder and R. Schrader, Axioms for Euclidean Green's functions. Commun. Math. Phys. 31, 83 (1973), 42, 281 (1975).
- [SCH] R. Schrader, Local operator products and field equations in $P(\phi)_2$ theories. Fortschr. Phys. 22, 611 (1974).
- [S I] T. Spencer, The mass gap for the $P(\phi)_2$ field model with a strong external field. Commun. Math. Phys. 39, 63 (1974).

- [S II] T. Spencer, The decay of the Bethe-Salpeter kernel in $P(\phi)_2$ quantum field models.
Commun. Math. Phys. 44, 143 (1975).
- [SZ] T. Spencer and F. Zirilli, Scattering states and bound states in $\lambda P(\phi)_2$. Commun. Math. Phys. 49, 1 (1976).
- [SY] K. Symanzik, On the many-particle structure of Green's functions in quantum field theory.
J. Math. Phys. 1, 249 (1960).
- [T] F. Trèves, Topological vector spaces, distributions and kernels. Academic Press, New York - London (1967).
- [GJ II] J. Glimm and A. Jaffe, The resummation of one particle lines. Preprint.

